Mod 2 Cohomology of Combinatorial Grassmannians

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Oriented matroids have long been of use in various areas of combinatorics [BLS+93]. Gelfand and MacPherson [GM92] initiated the use of oriented matroids in manifold and bundle theory, using them to formulate a combinatorial formula for the rational Pontrjagin classes of a differentiable manifold. MacPherson [Mac93] abstracted this into a manifold theory (*combinatorial differential (CD) manifolds*) and a bundle theory (which we call *combinatorial vector bundles* or *matroid bundles*). In this paper we explore the relationship between combinatorial vector bundles and real vector bundles. As a consequence of our results we get theorems relating the topology of the *combinatorial Grassmannians* to that of their real analogs.

The theory of oriented matroids gives a combinatorial abstraction of linear algebra; a k-dimensional subspace of $\mathbb{R}^n$ determines a rank k oriented matroid with elements $\{1, 2, \ldots, n\}$. Such oriented matroids can be given a partial order by using the notion of *weak maps*, which geometrically corresponds to moving the k-plane into more special position with respect to the standard basis of $\mathbb{R}^n$. The poset $\mathrm{MacP}(k, n)$ of rank k oriented matroids with n elements was defined by MacPherson in [Mac93] and is often called the *MacPhersonian*. The limit of the finite MacPhersonians gives an infinite poset $\mathrm{MacP}(k, \infty)$, and its geometric realization $\|\mathrm{MacP}(k, \infty)\|$ is the classifying space for rank k matroid bundles.

Our main results are the Combinatorialization Theorem, which associates a matroid bundle to a vector bundle, the Spherical Quasifibration Theorem, which associates a spherical quasifibration to a matroid bundle, and the Comparison Theorem, which shows that the composition of these two associations is essentially the forgetful functor. More precisely, if B is a regular cell complex, and if $V_k(B)$, $M_k(B)$, and $Q_k(B)$ denote the isomorphism classes of rank k vector bundles, matroid bundles, and spherical quasifibrations respectively, we construct maps

$$V_k(B) \to M_k(B) \to Q_k(B)$$

whose composite is the forgetful map given by deletion of the zero section. The first map is intimately related to the construction of a continuous map

$$\tilde{\mu} : G(k, \mathbb{R}^\infty) \to \|\mathrm{MacP}(k, \infty)\|$$

and the Comparison Theorem leads to the following theorem.

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Theorem A. *The map*

$$\tilde{\mu}^* : H^*(\|\operatorname{MacP}(k,\infty)\|;\mathbb{Z}_2) \to H^*(G(k,\mathbb{R}^\infty);\mathbb{Z}_2)$$

is a split surjection.

The mod 2 cohomology of $G(k,\mathbb{R}^\infty)$ is well-known: it is a polynomial algebra on the Stiefel-Whitney classes $w_1, w_2, \ldots, w_k$. The above theorem gives Stiefel-Whitney characteristic classes for matroid bundles.

More generally, one can consider any rank n oriented matroid M^n as a combinatorial analog to $\mathbb{R}^n$. There is an associated *combinatorial Grassmannian* $\Gamma(k, M^n)$ which is a partially ordered set of rank k "subspaces" of M^n. The MacPhersonian arises in this way as $\operatorname{MacP}(k,n) = \Gamma(k, M_c)$, where M_c is the unique rank n oriented matroid with elements $\{1, 2, \ldots, n\}$. If M^n is *realizable*, any realization induces a simplicial map

$$\tilde{\mu} : G(k,\mathbb{R}^n) \to \|\Gamma(k, M^n)\|$$

from a triangulation of the real Grassmannian of k-planes in $\mathbb{R}^n$ to the geometric realization of the combinatorial Grassmannian, constructed by the same method as for the special case of the MacPhersonian.

Theorem B. *The map*

$$\tilde{\mu}^* : H^*(\|\Gamma(k, M^n)\|;\mathbb{Z}_2) \to H^*(G(k,\mathbb{R}^n);\mathbb{Z}_2)$$

is a split surjection.

There is a natural combinatorial analog to an orientation of a real vector space leading to the definition of an *oriented combinatorial Grassmannian* $\tilde{\Gamma}(k, M^n)$ analogous to the Grassmannian $\tilde{G}(k,\mathbb{R}^n)$ of oriented k-planes in $\mathbb{R}^n$. For any realizable oriented matroid M^n, there is a combinatorialization map $\tilde{\mu}$ from $\tilde{G}(k,\mathbb{R}^n)$ to $\|\tilde{\Gamma}(k, M^n)\|$.

Theorem C. *The map*

$$\tilde{\mu}^* : H^*(\|\tilde{\Gamma}(k, M^n)\|;\mathbb{Z}) \to H^*(\tilde{G}(k,\mathbb{R}^n);\mathbb{Z})$$

has the Euler class in its image.

These results suggest substantial power for a combinatorial approach to characteristic classes via matroid bundles.

The Comparison Theorem also leads to results on homotopy groups of the combinatorial Grassmannian. We show that the second homotopy group of the MacPhersonian is the same as that of the corresponding Grassmannian. (Similar results for the zero and first homotopy groups of $\operatorname{MacP}(k,n)$ were previously known.). In addition, we get results on the homotopy groups of general combinatorial Grassmannians (which need not even be connected ([MRG93])).

Theorem D. *Let M^n be a realized rank k oriented matroid on n elements. Let p be a point in the image of $\tilde{\mu} : G(k,\mathbb{R}^n) \to \|\Gamma(k, M^n)\|$.*

1. *$\pi_k(\|\Gamma(k, M^n)\|, p)$ has $\mathbb{Z}$ as a subgroup when k is even and $n \geq 2k$.*

2. *$\pi_i(\|\Gamma(k, M^n)\|, p)$ has $\mathbb{Z}_2$ as a subquotient when $i \equiv 1, 2$ (mod 8), $n - k \geq i$, and $k \geq i$.*

3. *$\pi_{4m}(\|\Gamma(k, M^n)\|, p)$ has $\mathbb{Z}_{a_m}$ as a subquotient when $m > 0$, $n - k \geq 4m$, and $k \geq 4m$. Here a_m is the denominator of $B_m/4m$ expressed as a fraction in lowest terms, and B_m is the m-th Bernoulli number.*

Together with a previously known stability result ([And98]), this implies a similar statement about the infinite MacPhersonian. These are the first results giving nontrivial $\pi_i(\|\Gamma(k, M)\|, p)$ for general realizable M for any i.

Our results have potential interest in combinatorics and topology. A major focus of work on oriented matroids has been construction of oriented matroids behaving very differently from those in the image of $\tilde{\mu}$, but our partial computations of cohomology and homotopy groups should be viewed as an attempt to tame the beast. As far as geometric topology is concerned, a matroid bundle over a finite cell complex is a purely finite gadget, and we have shown that matroid bundles have characteristic class information. MacPherson has conjectured that in fact all characteristic classes of a vector bundle should be carried by the associated matroid bundle. Also, the study of matroid bundles is a necessary first step in the study of CD manifolds.

The paper is organized as follows. We review the theory of oriented matroids and develop the foundations of matroid bundles. We show that any vector bundle whose base space is a regular cell complex defines an isomorphism class of matroid bundles, thus passing from topology to combinatorics. We then construct a combinatorial "sphere bundle" (actually, a spherical quasifibration) associated to any matroid bundle, and thus pass from combinatorics back to topology. This construction is described in Section 2.3. This Combinatorialization Theorem depends on a deep result from combinatorics (the Topological Realization Theorem) and a deep result from topology (Quillen's Theorem B). From this construction we derive Stiefel-Whitney classes and Euler classes for matroid bundles. The Comparison Theorem (see Section 5) is proven by constructing a map of spherical quasifibrations from the canonical sphere bundle over the real Grassmannian to the "sphere bundle" over the combinatorial Grassmannian, which implies that the Stiefel-Whitney and Euler classes we defined for matroid bundles map under $\tilde{\mu}^*$ to the analogous classes for real vector bundles. This completes the proof of the Theorems A, B, and C. Theorem D (Section 6) is a consequence of the Combinatorialization Theorem, the Spherical Quasifibration Theorem, and the Comparison Theorem and homotopy theoretic results on the image of the J-homomorphism.

We also show that these combinatorial characteristic classes have interpretations analogous to those of their classical counterparts, as obstructions to the existence of combinatorial "orientations" (Theorem 4.8) and combinatorial "independent sets of vector fields" (Section 7).

We outline below the organization of the paper.

1 Preliminaries

1.1 Oriented Matroids

For introductions to oriented matroids, see [BLS+93] and [Mac93]. There are several equivalent axiomatizations of oriented matroids; throughout this paper we will use the *covector axioms* [BLS+93, p.159].

Definition 1.1. An **oriented matroid** M is a finite set $E(M)$ and a subset

$$\mathcal{V}^*(M) \subseteq \{-,0,+\}^{E(M)}$$

satisfying the following axioms:

1. $0 \in \mathcal{V}^*(M)$.
2. If $X \in \mathcal{V}^*(M)$, then $-X \in \mathcal{V}^*(M)$.
3. (*Composition*) If $X, Y \in \mathcal{V}^*(M)$, then the function

$$\begin{array}{rccl} X \circ Y: & E(M) & \to & \{-,0,+\} \\ & e & \mapsto & \begin{cases} X(e) & \text{if } X(e) \neq 0 \\ Y(e) & \text{otherwise} \end{cases} \end{array}$$

   is in $\mathcal{V}^*(M)$.
4. (*Elimination*) If $X(e) = +$ and $Y(e) = -$, then there is a $Z \in \mathcal{V}^*(M)$ such that $Z(e) = 0$, and for all $f \in E(M)$ for which $X(f)$ and $Y(f)$ are not of opposite signs, $Z(f) = X \circ Y(f)$.

$E(M)$ is called the set of **elements** of M. $\mathcal{V}^*(M)$ is the set of **covectors** of M.

The motivating example: consider n linear forms $\{\phi_1, \ldots, \phi_n\}$ on a finite dimensional real vector space V. To any $p \in V$ we associate a sign vector $X(p) = (\operatorname{sign}\phi_1(p), \ldots, \operatorname{sign}\phi_n(p)) \in \{-, 0, +\}^n$. The set $\{X(p) : p \in V\}$ is the set of covectors of an oriented matroid M with elements $\{1, 2, \ldots, n\}$. The set $\{\phi_1, \ldots, \phi_n\}$ is called a **realization** of M. Any M arising in this way is called **realizable**. Note that any of the ϕ_i can be multiplied by a positive scalar without changing M. Thus we can represent a realizable oriented matroid by an arrangement of hyperplanes $\{\phi_i^{-1}(0)\}$ and a distinguished side $\phi_i^{-1}(\mathbb{R}^+)$ for each hyperplane. (If $\phi_i = 0$ then the corresponding "hyperplane" is the **degenerate hyperplane** V and the "distinguished side" is $\emptyset$.) By considering a form as the inner product with a vector, we arrive at yet another way of viewing an realizable oriented matroid: Take a finite collection $E = \{v_1, \ldots, v_n\}$ of vectors in a finite dimensional real inner product space V, then the functions given by $\{i \mapsto \operatorname{sign}(v \cdot v_i) : v \in V\}$ are the covectors of an oriented matroid.

Definition 1.2. Let M be an oriented matroid with elements E. A subset I of E is **independent** in M if for every $e \in I$, there is a covector X so that $X(e) \neq 0$, but $X(I \backslash \{e\}) = 0$. The **rank** of M is the maximal order of a set of independent elements of M.

If the oriented matroid arises from a set of vectors in a real inner product space, then the rank of the oriented matroid equals the dimension of the span of E.

Definition 1.3. [BLS+93, pp. 133-134] Let $A \subseteq E(M)$ where M is an oriented matroid. Define two oriented matroids whose elements are $E(M)\backslash A$.

1. The covectors of the **deletion** $M\backslash A$ are

$$\{X|_{E(M)\backslash A} : X \in \mathcal{V}^*(M)\}.$$

2. The covectors of the **contraction** M/A are

$$\{X|_{E(M)\backslash A} : X \in \mathcal{V}^*(M) \text{ so that } X(a) = 0 \text{ for all } a \in A\}.$$

For a realizable oriented matroid, the deletion is realized by forgetting the linear forms in A, while the contraction is realized by restricting the forms in $E(M)\backslash A$ to the intersection of the zero sets of the forms in A.

Oriented matroids are connected to topology and geometry by the *Topological Representation Theorem* of Folkman and Lawrence. This gives a way to associate a PL sphere to an oriented matroid. We will state a weak version of the theorem here. First we take a definition from [BLS+93, §5.1].

Definition 1.4. A **pseudosphere** S in S^{k-1} is the image of the equator S^{k-2} under a homeomorphism $h : S^{k-1} \to S^{k-1}$. An **oriented pseudosphere** S is a

pseudosphere together with a labeling S^+ and S^- of the connected components of $S^{k-1}\backslash S$. Here S^+ and S^- are called the **(open) sides of** S. A **signed arrangement of pseudospheres** is a finite multiset $\mathcal{A} = (S_e)_{e\in E}$, where for each e, S_e is an oriented pseudosphere of S^{k-1}, provided the following three conditions hold:

1. $S_A = \cap_{e\in A} S_e$ is homeomorphic to a sphere, for all subsets A of E.
2. If $S_A \not\subseteq S_e$, for $A \subseteq E$ and $e \in E$, then $S_A \cap S_e$ is an oriented pseudosphere in S_A with sides $S_A \cap S_e^+$ and $S_A \cap S_e^-$.
3. The intersection of a collection of closed sides is either a sphere or a ball.

$\mathcal{A}$ is **essential** if $\cap_{e\in E} S_e = \emptyset$.

For instance, if $\{\phi_1, \dots, \phi_n\}$ is a realization of M in $\mathbb{R}^k$, then the arrangement of equators $\{\phi_i^{-1}(0) \cap S^{k-1}\}_{i\in\{1,2,\dots,n\}}$ is a signed arrangement of pseudospheres. This arrangement is essential if and only if the rank of M is k.

Any essential signed arrangement of pseudospheres chops S^{k-1} into a regular cell complex. The Topological Realization Theorem shows that the cells of this complex give the nonzero covectors of an oriented matroid and that essentially all oriented matroids arise in this way.

Let $\mathcal{A} = (S_e)_{e\in E}$ be a essential signed arrangement of psuedospheres in S^{k-1}. Define a function

$$\sigma : S^{k-1} \to \{+, 0, -\}^E$$

by $\sigma(x)(e) = +, 0,$ or $-$ depending on whether $x \in S_e^+, S_e,$ or S_e^-. Then

$$\{\sigma^{-1}(X) : X \in \sigma(S^{k-1})\}$$

gives a regular cell decomposition of S^{k-1} and $\mathcal{L}(\mathcal{A}) = \sigma(S^{k-1}) \cup \{0\}$ is the set of covectors of an oriented matroid on E (see [BLS$^+$93, §5.1]).

A **loop** in an oriented matroid is an element e so that $X(e) = 0$ for all covectors X. An oriented matroid is **loopfree** if it has no loops.

Topological Representation Theorem. *(cf. [FL78], [BLS$^+$93])*

1. *If M is a loopfree, rank k oriented matroid on E, there is an essential signed arrangement $\mathcal{A} = (S_e)_{e\in E}$ of pseudospheres in S^{k-1} with $\mathcal{L}(\mathcal{A}) = \mathcal{V}^*(M)$.*
2. *If $\mathcal{A} = (S_e)_{e\in E}$ is an essential signed arrangement of pseudospheres in S^{k-1}, then $\mathcal{L}(\mathcal{A})$ is the set of covectors of a loopfree, rank k oriented matroid on E.*

1.2 Partially ordered sets

Oriented matroids and geometric topology are related via partially ordered sets, or *posets*. There are several partial orders associated to oriented matroids, where moving up in the partial order corresponds to some notion of moving into more general position. The transition from posets to geometric topology is through a functor called *geometric realization*.

Definition 1.5. We define three partial orders:

1. On $\{-, 0, +\}$: The only strict inequalities are $+ > 0$ and $- > 0$.

2. On $\{-, 0, +\}^E$: Define $X \geq Y$ if $X(e) \geq Y(e)$ for all $e \in E$.

3. On the set of oriented matroids on a set E: Define $M_1 \geq M_2$ if for every $X \in \mathcal{V}^*(M_2)$ there is some $Y \in \mathcal{V}^*(M_1)$ such that $Y \geq X$.

This last relation is sometimes described by saying M_2 is a **weak map image** of M_1, or that M_2 is a **specialization** of M_1. If $\{\phi_1, \ldots, \phi_n\}$ is a realization of a rank k oriented matroid M_1 and $\{\xi_1, \ldots, \xi_n\}$ is a realization of M_2, then there is a weak map $M_1 \rightsquigarrow M_2$ if and only if $\text{sign}(\phi_{i_1} \wedge \cdots \wedge \phi_{i_k}) \geq \text{sign}(\xi_{i_1} \wedge \cdots \wedge \xi_{i_k})$ for every $\{i_i, \ldots, i_k\} \subseteq \{1, \ldots, n\}$.

An (abstract) **simplicial complex** K is a collection of non-empty finite sets, closed under proper inclusion. The elements of K are called *simplices*. An i-simplex is an simplex with $i+1$ elements, and $K^i \subset K$ is the set of i-simplices. Note that a simplicial complex is a poset, with partial order given by inclusion. A **regular cell complex** B is a CW complex so that every cell e has a characteristic map $D^n \to \overline{e}$ which is a homeomorphism. The face lattice $\mathcal{F}(B)$ is the poset of closed cells of B, ordered by inclusion. A simplicial complex K determines a regular cell complex, which we denote by $\|K\|$. A **chain** in a poset P is a non-empty, finite, totally ordered subset of P. The **order complex** ΔP of a poset P is the simplicial complex whose simplices are the chains in P.

Thus there are functors

$$\text{Posets} \xrightarrow{\Delta} \text{Simplicial Complexes} \xrightarrow{\|\ \|} \text{Regular Cell Complexes} \xrightarrow{\mathcal{F}} \text{Posets}$$

For a poset P, we write $\|P\|$ for $\|\Delta P\|$, and call it the **geometric realization** of P. If P^{op} is the poset given by reversing the inequalities, then $\|P\| \cong \|P^{\text{op}}\|$. For a regular cell complex B, the simplicial complex given by $\Delta\mathcal{F}(B)$ is called the **barycentric subdivision** of B and there is a homeomorphism $B \cong \|\mathcal{F}(B)\|$ under which every closed cell σ of B maps homeomorphically to the subcomplex $\|\mathcal{F}(B)_{\leq\sigma}\|$. Here if p is an element of a poset P, then $P_{\leq p} = \{q \in P : q \leq p\}$. For a simplicial complex K, there is a *canonical* homeomorphism $\|K\| \cong \|\mathcal{F}(K)\|$.

Order Homotopy Lemma. *If $f, g : P \to Q$ are poset maps so that for all $p \in P$, $f(p) \geq g(p)$ then $\|f\|$ is homotopic to $\|g\|$.*

The proof is easy. Simply let $(1 > 0)$ be the poset with elements 1 and 0, and the only strict inequality being $1 > 0$. Then apply $\| \ \|$ to the poset map $h : P \times (1 > 0) \to Q$ defined by $h(p, 1) = f(p)$ and $h(p, 0) = g(p)$. A consequence of the lemma is that if P has a maximal or minimal element, then $\|P\|$ is contractible.

1.3 Combinatorial Grassmannians

The **MacPhersonian** $\mathrm{MacP}(k, n)$ is the poset of rank k oriented matroids on the set $\{1, 2, \ldots, n\}$, with $M_1 \geq M_2$ if there is a weak map $M_1 \rightsquigarrow M_2$. There is an obvious embedding $\mathrm{MacP}(k, n) \hookrightarrow \mathrm{MacP}(k, n + 1)$, by adding $n + 1$ as a loop to each oriented matroid. We identify $\mathrm{MacP}(k, n)$ with its image under this embedding and define $\mathrm{MacP}(k, \infty)$ to be the direct limit over n of the $\mathrm{MacP}(k, n)$. For any rank n real vector space W with a fixed basis $\{w_1, \ldots, w_n\}$ (and therefore a fixed inner product) there is a canonical function

$$\mu : G(k, W) \to \mathrm{MacP}(k, n)$$

given by intersecting each $V \in G(k, W)$ with the hyperplanes $\{w_1^\perp, \ldots, w_n^\perp\}$ and considering the corresponding oriented matroid $\mu(V)$. Equivalently one projects the basis of W onto V and thus obtains an oriented matroid. This function μ is definitely not continuous, but the point inverses give an interesting decomposition of the Grassmannian. (See [BLS$^+$93, Section 2.4] for more about this decomposition.)

For future reference, we record a generalization of the MacPhersonian.

Definition 1.6. There is a **strong map** $M_1 \to M_2$ if $\mathcal{V}^*(M_2) \subseteq \mathcal{V}^*(M_1)$.

For instance, if $\{\phi_1, \ldots, \phi_n\}$ is a realization of M in V and W is a subspace of V, then $\{\phi_1|_W, \ldots, \phi_n|_W\}$ is the realization of a strong map image of M.

If M is an oriented matroid, the **combinatorial Grassmannian** $\Gamma(k, M)$ is the poset of all rank k strong map images of M, with partial order given by weak maps. If M is the *coordinate* rank n oriented matroid, i.e., the unique rank n oriented matroid with elements $\{1, 2, \ldots, n\}$, then $\Gamma(k, M)$ is the MacPhersonian $\mathrm{MacP}(k, n)$.

Let M be a realizable rank n oriented matroid, and let $\{v_1, \ldots, v_m\} \subset \mathbb{R}^n$ realize M. Then, as above, there is a function $\mu : G(k, \mathbb{R}^n) \to \Gamma(k, M)$ given by intersecting each vector space $V \in G(k, \mathbb{R}^n)$ with the oriented hyperplanes $\{v_1^\perp, \ldots, v_m^\perp\}$ and taking the corresponding oriented matroid $\mu(V)$.

2 Matroid bundles

We will define a combinatorial vector bundle over a regular cell complex B, to be an assignment of an oriented matroid $M(e)$ to every cell e of B so that if f is a face of e then $M(e)$ weak maps to $M(f)$. To see intuitively how a combinatorial vector bundle is derived from a vector bundle over a finite regular cell complex

B, note that if we fix a metric for the bundle and a finite set S of sections, then for every element b of the base space, $\{s(b)\}_{s\in S}$ determines an oriented matroid M_b with elements S. If we choose such an S so that for every b, $\{s(b)\}_{s\in S}$ spans the fiber over b and if the function $b \mapsto M_b$ is constant on the interior of each cell of B, then these oriented matroids determine a combinatorial vector bundle structure on B. In this case we say S is *tame* with respect to B. We will show that tame sections exist (perhaps after a subdivision of the base space) for vector bundles over finite-dimensional regular cell complexes.

2.1 Matroid bundles and their morphisms

The following is a generalization of the definition in [Mac93].

Definition 2.1. A rank k **matroid bundle** $\xi = (B, \mathcal{M})$ is a poset B and a poset map $\mathcal{M} : B \to \mathrm{MacP}(k, \infty)$.

Definition 2.2. The **universal rank** k **matroid bundle** is

$$\gamma_k = (\mathrm{MacP}(k, \infty), \mathrm{Id})$$

Definition 2.3. A rank k **combinatorial vector bundle** $\xi = (B, \mathcal{M})$ is a piecewise-linear (PL) cell complex B and a poset map $\mathcal{M}$ from the set of cells of a PL subdivision of B, ordered by inclusion, to $\mathrm{MacP}(k, \infty)$. In other words a combinatorial vector bundle $(B, \mathcal{M})$ is a matroid bundle $(\mathcal{F}(B'), \mathcal{M})$ where $\mathcal{F}(B')$ is the poset of cells of a PL subdivision B' of B.

Note every regular cell complex can be given the structure of a PL space via a barycentric subdivision.

A matroid bundle $(B, \mathcal{M})$ gives a combinatorial vector bundle $(\|\Delta B\|, \mathcal{M}')$, where $\mathcal{M}'(b_0 < b_1 < \cdots < b_m) = \mathcal{M}(b_m)$. A combinatorial vector bundle $\xi = (B, \mathcal{M} : \mathcal{F}(B') \to \mathrm{MacP}(k, \infty))$ induces a combinatorial vector bundle $\xi' = (B, \mathcal{M}' : \mathcal{F}(B'') \to \mathrm{MacP}(k, \infty))$ for any PL subdivision B'' of B', by sending any cell σ of B'' to $\mathcal{M}(\delta(\sigma))$, where $\delta(\sigma)$ is the smallest cell of B' containing σ. Two combinatorial vector bundles over a PL cell complex B are **equivalent** if they are equivalent under the equivalence relation generated by PL subdivision. Two matroid bundles over a poset B are **equivalent** if the associated combinatorial vector bundles are equivalent. Clearly there is a bijective correspondence between equivalence classes of matroid bundles over a poset and combinatorial vector bundles over its geometric realization, and henceforth we blur the distinction.

Definition 2.4. If $(B_1, \mathcal{M}_1)$ and $(B_2, \mathcal{M}_2)$ are two matroid bundles, a **morphism** from $(B_1, \mathcal{M}_1)$ to $(B_2, \mathcal{M}_2)$ is a triple $(f, [C_f, \mathcal{M}_f])$, where f is a PL map from $\|B_1\|$ to $\|B_2\|$, and $[C_f, M_f]$ is an equivalence class of combinatorial vector bundles over the mapping cylinder of f, where $\mathcal{M}_f$ restricts to structures equivalent to $(B_i, \mathcal{M}_i)$ at either end. Such a morphism is a **morphism covering** f. A morphism covering the identity on $\|B\|$ is called a B**-isomorphism**. Clearly equivalent bundles over B are B-isomorphic.

Definition 2.5. Let B_1 be a poset, $\xi = (B_2, \mathcal{M})$ a matroid bundle, and $f : \|B_1\| \to \|B_2\|$ a PL map. Then f is simplicial with respect to some PL subdivisions B_1' and B_2' of $\|B_1\|$ and $\|B_2\|$. The composite map $\mathcal{F}(B_1') \to \mathcal{F}(B_2') \to \operatorname{MacP}(k, \infty)$ defines a combinatorial vector bundle over $\|B_1'\|$, and any subdivision of B_1' gives an equivalent bundle. Thus f defines an equivalence class of combinatorial vector bundles over $\|B_1\|$. We call any of the corresponding matroid bundles a **pullback** of ξ by f and denote it $f^*(\xi)$.

Definition 2.6. If $(B_i, \mathcal{M}_i)$, $i \in \{1,2,3\}$ are rank k matroid bundles and $(f : B_1 \to B_2, [C_f, \mathcal{M}_f])$ and $(g : B_2 \to B_3, [C_g, \mathcal{M}_g])$ are morphisms between them, then consider the space $C_f \cup_{B_2} C_g$ obtained from the disjoint union of C_f and C_g by identifying each $b \in B_2 \subset C_f$ with $b \times 0 \in C_g$. The matroid bundle structures on C_f and C_g define a matroid bundle structure on this space. The pullback of the PL map $c : C_{g\circ f} \to C_f \cup_{B_2} C_g$ defined by

$$c[b,i] = \begin{cases} [b, 2i] & \text{if } i \le 1/2 \\ [f(b), 2(i - \frac{1}{2})] & \text{if } i \ge 1/2 \end{cases}$$

and $c[b] = [b]$ for all $b \in B_3$, defines an equivalence class $[C_{g\circ f}, \mathcal{M}_{g\circ f}]$ of matroid bundles which we call the **composition** of the two original morphisms.

Note that two bundles are B-isomorphic if and only if there is a combinatorial vector bundle over $\|B\| \times I$ restricting on the ends to bundles equivalent to the original ones.

The following familiar properties of bundles are easily verified.

Proposition 2.7. *1. Let B_1 and B_2 be posets. There is a morphism $\xi_1 \to \xi_2$ covering a PL-map $f : \|B_1\| \to \|B_2\|$ if and only if $f^*\xi_2$ is B_1-isomorphic to ξ_1.*

2. If $\phi, \psi : A \to B$ are poset maps so that $\|\phi\|$ and $\|\psi\|$ are homotopic, and if ξ is a matroid bundle over B, then $\phi^*\xi$ is A-isomorphic to $\psi^*\xi$.

3. Every rank k matroid bundle is a pullback of the universal rank k bundle.

4. If ξ is a rank k matroid bundle and there are two morphisms $\xi \to \gamma_k$ covering f and g respectively, then f and g are homotopic.

Recall that $[X, Y]$ is the set of homotopy classes of maps from X to Y.

Corollary 2.8. *For a regular cell complex B, let $M_k(B)$ be the set of B-isomorphism classes of rank k combinatorial vector bundles over B. Then*

$$\begin{aligned} M_k(B) &\to [B, \|\operatorname{MacP}(k,\infty)\|] \\ [\xi] &\mapsto [\|\mathcal{M}(\xi)\|] \end{aligned}$$

is a bijection, natural in B. The inverse is defined by applying the simplicial approximation theorem to [f] to obtain a subdivision B' of B, a poset map $f' : B' \to \mathcal{F}\|\operatorname{MacP}(k,\infty)\|$, and thus a matroid bundle $(B', \mathcal{M})$ where $\mathcal{M}(\sigma)$ is the maximal vertex of $f'(\sigma)$.

Thus the classifying space for rank k combinatorial vector bundles is $\|\operatorname{MacP}(k,\infty)\|$ with universal element γ_k.

Matroid bundles arise in combinatorics in a variety of ways: see [And99a] for examples. In addition, any real vector bundle yields a combinatorial vector bundle, as described in the following section.

2.2 Combinatorializing vector bundles: the Combinatorialization Theorem

For any real rank k vector bundle $\xi = (p : E \to B)$ over a paracompact base space there is a bundle map

$$\begin{array}{ccc} E & \xrightarrow{\tilde{c}} & E(k,\mathbb{R}^\infty) \\ \big\downarrow & & \big\downarrow \\ B & \xrightarrow{c} & G(k,\mathbb{R}^\infty) \end{array}$$

to the canonical bundle over the Grassmannian of k-planes in $\mathbb{R}^\infty$, and c is determined up to homotopy (cf. [MS74] Ch. 5). If B is the underlying space of a regular cell complex, we call the map c **tame** if $\mu \circ c$ is constant on the interior of each cell. Such a tame classifying map gives a combinatorial vector bundle $c(\xi) = (\mathcal{F}(B), \mathcal{M})$ by defining $\mathcal{M}(\sigma) = \mu(c(\text{int } \sigma))$. Here $\mathcal{M}$ is a poset map since if σ is a face of τ , then

$$c(\text{int } \sigma) \cap \overline{c(\text{int } \tau)} \neq \emptyset$$

so $\mu(c(\text{int } \sigma)) \leq \mu(c(\text{int } \tau))$ by [BLS$^+$93, 2.4.6]. A subdivision of the cell complex leads to an equivalent combinatorial vector bundle.

Generalizing the notation a bit, for any vector space V, let $G(k,V)$ be the Grassmannian of k-planes in V. A **classifying map** for a vector bundle $\xi = (p : E \to B)$ is a map $c : B \to G(k,V)$ covered by a bundle map from ξ to the canonical bundle. For any finite set F, let $\operatorname{MacP}(k,F)$ be the poset of rank k oriented matroids with elements F. For any set A, let $\operatorname{MacP}(k,A)$ be the direct limit of $\operatorname{MacP}(k,F)$, taken over all finite subsets F of A. If V is a vector space with a finite basis A, define $\mu : G(k,V) \to \operatorname{MacP}(k,A)$ as we did in Section 1.3, while if the basis A is infinite, define μ so that it restricts to $\mu : G(k, \text{Span } F) \to \operatorname{MacP}(k,F)$ for all finite subsets F of A. If V is a vector space, a **tame classifying map** is a classifying map $c : B \to G(k,V)$ so that $\mu \circ c$ is constant on the interior of each cell.

Theorem 2.9 (Combinatorialization Theorem). *Let $\xi = (p : E \to B)$ be a rank k real vector bundle, where B is the underlying space of a regular cell complex.*

1. *For $i = 0, 1$, let $c_i : B \to G(k,V_i)$ be a tame classifying map for ξ. Then there is a tame classifying map $h : B \times I \to G(k, V_0 \oplus V_1)$ for $\xi \times I$, restricting to c_i on $B \times \{i\}$.*

2. *If B is finite dimensional, any classifying map $c : B \to G(k, V)$ is homotopic to a classifying map which is tame with respect to some simplicial subdivision of the barycentric subdivision of B.*

Proof. Note that for a vector bundle $\xi = (p : E \to B)$, specifying a classifying map $c : B \to G(k, V)$ together with a covering map $\tilde{c} : E \to E(k, V)$ is equivalent to specifying a map for $\hat{c} : E \to V$ which is linear and injective on each fiber. (Here $c(b) = \hat{c}(p^{-1}b)$.) If V has a basis A, then c is tame if and only if the function

$$\begin{aligned} B &\to \text{subsets of } \{+, -, 0\}^A \\ b &\mapsto \{a \mapsto \operatorname{sign}(\hat{c}(e) \cdot a)\}_{e \in p^{-1}b} \end{aligned}$$

is constant on the interior of cells.

We prove something slightly more general than (1). Suppose that V has a basis A and that $c_0, c_1 : B \to G(k, V)$ are two tame classifying maps for ξ with covering maps $\widehat{c_0}, \widehat{c_1}$. Suppose also that for every $e \in E$ and for every $a \in A$, $\widehat{c_0}(e) \cdot a$ and $\widehat{c_1}(e) \cdot a$ do not have opposite signs. Then

$$\widehat{h_t}(e) = (1 - t)\widehat{c_0}(e) + t\widehat{c_1}(e) \qquad 0 \le t \le 1$$

defines a classifying map $h : B \times I \to G(k, V)$ for the bundle $E \times I \to B \times I$ which is tame with respect to the product cell structure on $B \times I$ and hence a tame homotopy between c_0 and c_1. Applying this to $V = V_0 \oplus V_1$ gives (1).

For part (2), it suffices to prove it for the universal case. We will find a triangulation of $G(k, \mathbb{R}^n)$ so that the identity map is tame, i.e., $\mu : G(k, \mathbb{R}^n) \to \operatorname{MacP}(k, n)$ is constant on the interior of simplices. Then given a classifying map $c : B \to G(k, V)$ where B has dimension r, by the cellular approximation theorem and the Schubert cell decomposition of the Grassmannian, there is a homotopic map $c' : B \to G(k, V')$ where $V' \subseteq V$ is a vector space spanned by $k + r$ elements of the basis. Finally, apply the Simplicial Approximation Theorem to map from the barycentric subdivision of B to the tame triangulation of $G(k, V)$.

Thus the following lemma applied to the coordinate oriented matroid (where $\Gamma(k, M) = \operatorname{MacP}(k, n)$) completes the proof of the combinatorialization theorem.

Lemma 2.10. *(cf. [Mac93]) Let M be a realizable rank n oriented matroid with a fixed realization. Then there is a semi-algebraic triangulation T of $G(k, \mathbb{R}^n)$ and a simplicial map with respect to its barycentric subdivision*

$$\tilde{\mu} : G(k, \mathbb{R}^n) \to \|\Gamma(k, M)\|$$

such that for every vertex v in the barycentric subdivision, one has $\tilde{\mu}(v) = \mu(v)$. Furthermore, the homotopy class of $\tilde{\mu}$ is independent of the choice of semi-algebraic triangulation.

Proof. This is an application of Appendix A, theorems on existence and uniqueness of semi-algebraic triangulations, and the fact that (cf. 2.4.6 in [BLS+93]) $\mu : G(k, \mathbb{R}^n) \to \|\Gamma(k, M)\|$ is upper semi-continuous.

A key tool is [Hir75, *Semi-algebraic triangulation theorem*] which proves that for any finite partition $\{U_i\}_{i \in I}$ of a bounded, semi-algebraic set S into semi-algebraic sets there exists a semi-algebraic triangulation of S such that each U_i is a union of the interiors of simplices. Furthermore, by [Hir75, 2.4], for any two such semi-algebraic triangulations, there is a semi-algebraic triangulation which is a common refinement. Thus S is a PL space.

In the case at hand $\{\mu^{-1}(N)\}_{N \in \Gamma(k,M)}$ is a semi-algebraic partition of $G(k, \mathbb{R}^n)$. In the language of Corollary A.4, the corresponding triangulation refines the stratification given by the upper semi-continuous map μ, so the result follows. ◻

◻

Corollary 2.11. *Let B be a finite dimensional regular cell complex. Let $V_k(B)$ be the set of B-isomorphism classes of rank k vector bundles over B. There is a "combinatorialization map"*

$$C : V_k(B) \to M_k(B),$$

natural in B, defined by sending a vector bundle to the combinatorial vector bundle given by a tame classifying map.

Proof of Corollary. Let ξ be a k-dimensional vector bundle over B.

- *Existence*: The combinatorialization theorem shows there is a tame classifying map
$$c : B \to G(k, V).$$
Define $C[\xi]$ to be the corresponding combinatorial vector bundle $[c(\xi)]$.

- *Uniqueness*: Suppose
$$c_i : B \to G(k, V_i) \qquad i = 0, 1$$
are two tame classifying maps. Applying the first part of the combinatorialization theorem gives a tame classifying map
$$h : B \times I \to G(k, V_0 \oplus V_1),$$
for $\xi \times I$ restricting to c_0 and c_1 at either end. The resulting combinatorial vector bundle over $B \times I$ gives an B-isomorphism between $c_0(\xi)$ and $c_1(\xi)$.

- *Naturality*: Clear.

◻

We wish to extend the map $V_k(B) \to M_k(B)$ to infinite dimensional complexes. The problem, as shown in [AD], is that $G(k, \mathbb{R}^\infty)$ has no tame triangulation, i.e. no triangulation where μ is constant on simplices. But we do have the following theorem.

Theorem 2.12. *There is a map $\tilde{\mu} : G(k, \mathbb{R}^\infty) \to \|\operatorname{MacP}(k, \infty)\|$ which restricts to a map $G(k, \mathbb{R}^n) \to \|\operatorname{MacP}(k, n)\|$ given by Lemma 2.10 for all n. The homotopy class of $\tilde{\mu}$ is well-defined.*

Proof. This follows from Appendix A and results on existence of semi-algebraic triangulations, by the same argument as the proof of Lemma 2.10. □

Corollary 2.13. *Let B be a regular cell complex. There is a "combinatorialization map"*

$$C : V_k(B) \to M_k(B),$$

natural in B, which for finite-dimensional B coincides with the map given by sending a vector bundle to the combinatorial vector bundle given by a tame classifying map.

Proof. By replacing B by $\|\mathcal{F}(B)\|$, we may assume that B is the geometric realization of a simplicial complex. Let $c : B \to G(k, \mathbb{R}^\infty)$ be a classifying map for ξ. Apply the simplicial approximation theorem to $\tilde{\mu} \circ c$ to find a subdivision B' of B and a map $c' : \mathcal{F}(B') \to \operatorname{MacP}(k, \infty)$, where $\|c'\|$ is homotopic to $\tilde{\mu} \circ c$. Then set $C[\xi] = [c']$. □

2.3 Combinatorial sphere and disk bundles

Gelfand and MacPherson, in their combinatorial formula for the Pontrjagin classes of a differentiable manifold, constructed a "combinatorial sphere bundle" associated to a matroid bundle:

Definition 2.14. For a matroid bundle $\xi = (B, \mathcal{M})$, define posets

$$E(\xi) = \{(\sigma, X) : \sigma \in B,\ X \in \mathcal{V}^*(\mathcal{M}(\sigma))\ \}$$
$$E_0(\xi) = \{(\sigma, X) : \sigma \in B,\ X \in \mathcal{V}^*(\mathcal{M}(\sigma)) \backslash \{0\}\ \}$$

with $(\sigma, X) \geq (\sigma', X')$ if $\sigma \geq \sigma'$ and $X \geq X'$.

The projection map $\pi_0 : E_0(\xi) \to B$ and $\pi : E(\xi) \to B$ are the **combinatorial sphere bundle** and **combinatorial disk bundle** associated to ξ.

Example 2.15. *Warning: The geometric realization of the combinatorial sphere bundle may not be a topological sphere bundle!* The Topological Representation Theorem promises that the realization of each fiber of π_0 over a vertex is a PL sphere. But in general, the realization of π_0 is *not* a topological sphere bundle, as we can see from the example in Figure 1.

This figure shows a weak map $M_1 \rightsquigarrow M_0$ of rank 2 (realizable) oriented matroids with elements $\{a, b, c\}$; in the second oriented matroid the element b

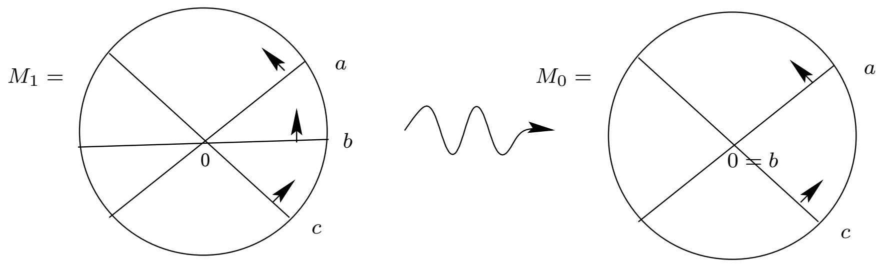

Figure 1: A sphere bundle need not be a sphere bundle.

is the degenerate hyperplane $0^\perp$. For each oriented matroid, the nonzero covectors are given by the cell decomposition of the unit circle. We can define a rank 2 matroid bundle over the poset $(1 > 0)$ by sending 1 to M_1 and 0 to M_0. The total space of the associated sphere bundle will contain a 3-simplex $\{(1, a^-b^+c^+), (1, a^-b^0c^+), (0, a^-b^0c^+), (0, a^0b^0c^+)\}$. Hence the geometric realization of the combinatorial sphere bundle is not a topological circle bundle over the 1-dimensional cell $\|1 > 0\|$.

It seems unlikely that the geometric realization of π_0 always gives a fibration. In Section 3 we will show that it is the next best thing, a spherical quasifibration.

3 Combinatorial bundles are quasifibrations

In this section we review the notion (due to Dold-Thom [DT56]) of a quasifibration and show that the combinatorial sphere bundle associated to a matroid bundle is a spherical quasifibration. A key tool is a criterion for the geometric realization of a poset map to be a quasifibration. This criterion was formulated in the Ph.D. thesis [Bab93] of Eric Babson. We give a proof of Babson's criterion in Appendix B. It is an application of Quillen's work on the foundations of algebraic K-theory.

3.1 Quasifibrations

Definition 3.1. A map $p : E \to B$ is a **fibration** if it has the homotopy lifting property [Whi78]. A map $p : E \to B$ is a **quasifibration** if

$$p_* : \pi_i(E, p^{-1}b, e) \to \pi_i(B, \{b\}, b)$$

is an isomorphism for all $i \geq 0$, for all $b \in B$, and for all $e \in p^{-1}b$.

Definition 3.2. A **spherical (quasi)-fibration of rank** k is a (quasi)-fibration $p_0 : E_0 \to B$ so that for all $b \in B$, $p_0^{-1}b$ has the weak homotopy type of S^{k-1} (i.e., there is a map $S^{k-1} \to p_0^{-1}b$ inducing an isomorphism on homotopy groups).

A fibration is a quasifibration. An example of a quasifibration which is not a fibration is given by collapsing a closed subinterval of an interval to a point. A (quasi)-fibration has a long exact sequence in homotopy.

A construction of Bourbaki [Whi78, §I.7] shows that every continuous map $f : E \to B$ has the homotopy type of a fibration, i.e. there is a fibration $\pi_f : P_f \to B$ and a homotopy equivalence $h : E \to P_f$ so that $f = \pi_f \circ h$. Here

$$P_f = \{(e, \alpha) \in E \times B^I : f(e) = \alpha(0)\}$$

$\pi_f(e, \alpha) = \alpha(1)$ and $h(e) = (e, \mathrm{const}_e)$. For $b \in B$, $f^{-1}b$ is the **fiber above** b and $\pi_f^{-1}b$ is the **homotopy fiber above** p. The homotopy long exact sequence of a fibration and the five lemma give the following alternative (and perhaps better) definition of a quasifibration.

Proposition 3.3. *A map $p : E \to B$ is a quasifibration if and only if for all $b \in B$*

$$p^{-1}(b) \to {\pi_p}^{-1}(b)$$

is a weak homotopy equivalence.

Definition 3.4. A **morphism of (quasi)-fibrations** p and p' is a map $f : B \to B'$ and a (quasi)-fibration $E_f \to C_f$ over the mapping cylinder of f which restricts on the ends to f and f'. Such a morphism is called a **morphism covering** f. A morphism covering the identity on B is called a B**-isomorphism**. A **CW-(quasi)-fibration**, respectively a B**-CW-isomorphism**, is a (quasi)-fibration, respectively B-isomorphism, in which the domain of each (quasi)-fibration has the homotopy type of a CW-complex.

Let $p' : E' \to B$ and $p : E \to B$ be two maps. A map $g : E' \to E$ is **fiber-preserving** if $p \circ g = p'$. A **fiber-preserving homotopy equivalence** (f.p.h.e) is a homotopy equivalence $g : E' \to E$ which is fiber-preserving. There is also the notion of a **fiber-preserving weak homotopy equivalence** (f.p.w.h.e). Two maps $g, g' : E' \to E$ are **fiberwise homotopic** if there is a homotopy $G : E' \times I \to E$ between them so that for all t, $G(-, t)$ is fiber-preserving. A fiber-preserving map $g : E' \to E$ is a **fiber homotopy equivalence** (f.h.e) if there is a fiber-preserving map $h : E \to E'$ so that $g \circ h$ and $h \circ g$ are both fiberwise homotopic to the identity. One says that p and p' have the same fiber homotopy type. Of course a fiber homotopy equivalence is a fiber-preserving homotopy equivalence.

Two quasifibrations $p : E \to B$ and $p' : E' \to B$ are B-isomorphic if and only if they are equivalent under the equivalence relation generated by f.p.w.h.e. Indeed if $g : E' \to E$ is a f.p.w.h.e., then the natural map $C_g \to B \times I$ shows that p and p' are B-isomorphic quasifibrations. Conversely given a B-isomorphism $p'' : E'' \to B \times I$, the inclusion map gives a f.p.w.h.e. from p (or p') to $\mathrm{pr}_B \circ p'' : E'' \to B$.

Two fibrations $p : E \to B$ and $p' : E' \to B$ are B-isomorphic if and only if there is a fiber homotopy equivalence between them, which occurs if and only if

there is a fiber preserving homotopy equivalence between them. All of this is an elementary, if somewhat confusing, exercise in the homotopy lifting property. For a reference that B-isomorphism implies f.h.e., see [Whi78, Theorem 7.25] and for a reference that f.p.h.e. implies h.e., see [Dol63, Theorem 6.1].

The following theorem shows that in terms of homotopy theory, there is really not much difference between fibrations and quasifibrations. This theorem, in slightly different language, is due to Stasheff [Sta63].

Theorem 3.5. *For a CW-complex B, let $Q(B)$ (respectively $F(B)$) be the set of B-CW-isomorphism classes of CW-quasifibrations (respectively CW-fibrations) over B. There is a bijection,*

$$Q(B) \to F(B),$$

given by converting a quasifibration p into a fibration π_p. The inverse is the forgetful map, given by considering a fibration as a quasifibration.

Proof. This conversion process has two nice properties. The first is that it sends a fiber-preserving homotopy equivalence to a fiber-preserving homotopy equivalence. The second is that given a map $p : E \to B$ where E and B have the homotopy type of a CW-complex, then P_p also has the homotopy type of a CW-complex [Mil59].

We need to see that the map $Q(B) \to F(B)$ is well-defined. If $p'' : E'' \to B \times I$ is a B-isomorphism between quasifibrations $p : E \to B$ and $p' : E' \to B$, then as above, there is a a f.p.w.h.e. from p (or p') to $\mathrm{pr}_B \circ p''$, which is a f.p.h.e by the CW-assumption. Now this conversion process takes a f.p.h.e to a f.p.h.e, and hence the corresponding fibrations are B-CW-isomorphic.

If one first converts and then forgets, one obtains a quasifibration which is f.p.h.e. to the original one, and hence equivalent. Conversely, if one has a fibration and converts it, the result is a fibration equivalent to the original one. $\square$

Remark 3.6. By the pullback, $F(-)$ is a contravariant functor from topological spaces to sets. However, since the pullback of a quasifibration need not be a quasifibration, it is not clear that $Q(-)$ is a functor. However, using the equivalence in the above theorem, $Q(-)$ does give a functor from the category of spaces having the homotopy type of CW-complexes to sets. Furthermore, if the pullback of a quasifibration $p : E \to B$ under a map $f : B' \to B$ happens to be a quasifibration, then the pullback $f^*E \to B'$ represents the correct induced element of $Q(B')$.

3.2 The spherical quasifibration theorem

Spherical Quasifibration Theorem. *For any matroid bundle $\xi = (B, \mathcal{M})$, the geometric realizations of the combinatorial sphere and disk bundles*

$$\|\pi_0\| : \|E_0(\xi)\| \to \|B\|$$
$$\|\pi\| : \|E(\xi)\| \to \|B\|$$

are quasifibrations.

We use the following criterion for the geometric realization of a poset map to be a quasifibration.

Babson's Criterion. *If $f : P \to Q$ is a poset map satisfying both of the conditions below, then $\|f\|$ is a quasifibration.*

1. $\|f^{-1}q \cap P_{\leq p}\|$ *is contractible whenever $p \in P$, $q \in Q$, and $q \leq f(p)$.*
2. $\|f^{-1}q \cap P_{\geq p}\|$ *is contractible whenever $p \in P$, $q \in Q$, and $q \geq f(p)$.*

This criterion was formulated in the Ph.D. thesis [Bab93] of Eric Babson, and is an application of Quillen's work on the foundations of algebraic K-theory. We give a proof in Appendix B. In this section we verify that the combinatorial bundles satisfy Babson's criterion.

Lemma 3.7. *If $M \rightsquigarrow M'$ and e is a nonzero element of M' then $M/e \rightsquigarrow M'/e$.*

Proof. Let $X' \in \mathcal{V}^*(M'/e) = \{Z' \in \mathcal{V}^*(M') \ : \ Z'(e) = 0\}$. Since e is nonzero in M', there are covectors Z_1' and Z_2' of M' so that $Z_1'(e) = +$ and $Z_2'(e) = -$. Since $M \rightsquigarrow M'$, there are covectors X_1 and X_2 of M so that $X_1 \geq X' \circ Z_1'$ and $X_2 \geq X' \circ Z_2'$. Since $X_1(e) = +$ and $X_2(e) = -$, we can apply the elimination axiom in the definition of an oriented matroid. What results is a covector X of M/e so that $X \geq X'$. □

Lemma 3.8. *If $M \rightsquigarrow M'$,* rank $M =$ rank M', *and X is a nonzero covector of M, then there is a nonzero covector X' of M' so that $X \geq X'$.*

Proof. We induct on rank(M) and on the number of elements of M. When rank$(M) = 1$, the existence of X' is easy.

If rank$(M) > 1$, it suffices to consider the case when X is not maximal, since if X is maximal, there is a nonzero covector $\overline{X}$ of M so that $X > \overline{X}$ and we replace X by $\overline{X}$. So assume that there is some nonzero element e of M such that $X(e) = 0$. We have two cases:

- If e is zero in M', then $M \backslash e \rightsquigarrow M' \backslash e$. Then by induction on the number of elements we get $X' \in \mathcal{V}^*(M' \backslash e) = \mathcal{V}^*(M')$ such that $X \geq X'$.
- If e is nonzero in M', then by Lemma 3.7 we have $M/e \rightsquigarrow M'/e$. Since $X \in \mathcal{V}^*(M/e)$, induction on rank gives a nonzero $X' \in \mathcal{V}^*(M'/e) \subset \mathcal{V}^*(M')$ such that $X \geq X'$.

□

Remark 3.9. A consequence of this lemma is that if $M \rightsquigarrow M'$ and rank $M =$ rank M', there is a poset map $\Phi : \mathcal{V}^*(M) \to \mathcal{V}^*(M')$ which maps nonzero covectors to nonzero covectors and so that $X \geq \Phi(X)$ for all X. Indeed, $\Phi(X)$ is defined to be the composition of all nonzero covectors of M' which are less than or equal to X. This map Φ lends credence to the intuition that a weak map corresponds to moving into special position. This map and variations are explored further in [And].

Lemma 3.10. *Let K be a simplicial decomposition of a compact PL manifold with boundary, let $K^0 = \{\sigma \in K : \|\sigma\| \cap \|\partial K\| = \emptyset\}$, and assume that ∂K is* ***full****, i.e., any simplex whose faces are all contained in ∂K is itself contained in ∂K. Then $\|K\| \simeq \|K^0\|$.*

Proof. Enumerate the simplices $\sigma_1, \sigma_2, \ldots, \sigma_k$ of ∂K so that the dimension is monotone decreasing. Let

$$K_i = K \backslash (K_{\geq \sigma_1} \cup \cdots \cup K_{\geq \sigma_{i-1}}).$$

Note $K_1 = K$ and $K_{k+1} = K^0$.

We show $\|K^0\|$ can be obtained from $\|K\|$ by a sequence of elementary collapses and expansions. Define an elementary collapse to be **inward** if it collapses out a pair of simplices ω and $\omega \cup \{x\}$ with $\{x\} \notin \partial K$. (We will use the term inward to apply to collapses of any complex, not just K.) We show by induction on dimension and induction on i that $\|K_{i+1}\|$ can be obtained from $\|K_i\|$ by a sequence of elementary collapses and expansions. Both initial cases hold because ∂K is full.

We will use $\sim$ to denote equivalent via a sequence of elementary collapses inwards and elementary expansions.

$$\begin{aligned}
\|\operatorname*{link}_{K_i} \sigma_i\| &\sim \|\operatorname*{link}_{K} \sigma_i\| && \text{(induction on } i \text{ and } K \sim K_i)\\
&\sim \|\operatorname*{link}_{K^0} \sigma_i\| && \text{(induction on dimension and } \|\operatorname*{link}_{K} \sigma_i\| \text{ is a PL-ball)}\\
&\sim * && \text{(contractible by the last step and } \sim * \text{ since } K^0 \cap \partial K = \emptyset)
\end{aligned}$$

We leave for the reader to verify that such a sequence of collapses inward and expansions from $\|\operatorname{link}_{K_i} \sigma_i\|$ to a point gives a sequence of collapses inward and expansions from $\|\operatorname{star}_{K_i} \sigma_i\| = \sigma_i * \|\operatorname{link}_{K_i} \sigma_i\|$ to $\|\operatorname{link}_{K_i} \sigma_i\|$, and hence from $\|K_i\|$ to $\|K_{i+1}\|$.

□

Proof of Spherical Quasifibration Theorem. We apply Babson's Criterion, first with $P = E_0(\xi)$ and $Q = B$, then with $P = E(\xi)$ and $Q = B$. In the first case, let $(M, X) \in E_0(\xi)$ and $M' \in B$. Then $X \in \mathcal{V}^*(M) \backslash \{0\}$ and $\pi^{-1}(M') \cong \mathcal{V}^*(M') \backslash \{0\}$.

If $M \rightsquigarrow M'$, then $\pi^{-1}(M') \cap E_0(\xi)_{\leq (M,X)}$ is isomorphic to the poset of all covectors X' of M' such that $X \geq X'$. This is the poset of all covectors in M' corresponding to cells of

$$B^X_{M'} = \Big(\bigcap_{e \in X^{-1}(+)} \overline{S_e^+}\Big) \cap \Big(\bigcap_{e \in X^{-1}(-)} \overline{S_e^-}\Big) \cap \Big(\bigcap_{e \in X^{-1}(0)} S_e\Big)$$

given as a subcomplex of the pseudosphere picture of M'. By the Topological Representation Theorem, this is either empty, a PL-sphere, or a PL-ball. It can't be a sphere since $X \neq 0$ and it is non-empty by Lemma 3.8. Thus the first condition of Babson's Criterion is fulfilled.

If $M' \rightsquigarrow M$, then $\pi^{-1}(M') \cap E_0(\xi)_{\geq (M,X)}$ is isomorphic to the poset of all covectors X' of M' such that $X' \geq X$. This is the poset of all covectors in M' corresponding to cells in the interior of the cell complex

$$B_X^{M'} = (\bigcap_{e \in X^{-1}(+)} \overline{S_e^+}) \cap (\bigcap_{e \in X^{-1}(-)} \overline{S_e^-})$$

given as a subcomplex of the pseudosphere picture of M'. Now $B_X^{M'}$ must be empty or a PL-ball, but is in fact a PL-ball of full rank since this is a non-empty intersection (containing cells corresponding to covectors $X' \geq X > 0$). By the previous lemma, the realization of the poset of cells in the interior of this ball is contractible. Thus the second condition of Babson's Criterion is fulfilled, and so the realization of the combinatorial sphere bundle is a quasifibration.

In the case of the disk bundle, the first condition to check is trivial, since the poset in question will have a unique minimal element. The second condition follows immediately from the proof of the second condition for sphere bundles. □

Corollary 3.11. *Let B be a regular cell complex and $Q_k(B)$ be the set of B-isomorphism classes of rank k spherical quasifibrations over B. The geometric realization of the combinatorial sphere bundle gives a well-defined map*

$$\|E_0\| : M_k(B) \to Q_k(B)$$

natural in B.

We now have two maps $V_k(B) \to Q_k(B)$, the map above and the map given by deleting the zero section of a vector bundle. In Section 5 we will show they coincide.

4 Stiefel-Whitney classes and Euler classes of matroid bundles

Recall the axioms for Stiefel-Whitney classes [MS74, §4]:

1. For any vector bundle $\xi = (p : E \to B)$ there are classes $w_i(\xi) \in H^i(B; \mathbb{Z}_2)$, with $w_0(\xi) = 1$ and $w_i(\xi) = 0$ when i is larger than the fiber dimension.
2. If $f : B' \to B$ is covered by a bundle map $\xi' \to \xi$, then $w_i(\xi') = f^* w_i(\xi)$.
3. $w_n(\xi_0 \oplus \xi_1) = \sum_{i=0}^{n} w_i(\xi_0) \cup w_{n-i}(\xi_1)$.
4. The first Stiefel-Whitney class of the canonical line bundle over $\mathbb{R}P^\infty$ is non-trivial.

The construction of the Stiefel-Whitney classes of a vector bundle $\xi = (p : E \to B)$ with fiber $\mathbb{R}^k$ given in [MS74, §8] is

$$w_i(\xi) = \phi^{-1}\mathrm{Sq}^i\phi(1) \in H^i(B; \mathbb{Z}_2)$$

where Sq^i is the i-th Steenrod square and

$$\phi : H^*(B; \mathbb{Z}_2) \to H^{*+k}(E, E_0; \mathbb{Z}_2)$$

is the Thom isomorphism. We next review the construction of Stiefel-Whitney classes and Euler classes for spherical (quasi)-fibrations.

Thom Isomorphism Theorem. *Let $p_0 : E_0 \to B$ be a rank k spherical quasifibration. Let $p : E \to B$ be a quasifibration with contractible fiber and a fiber-preserving embedding $E_0 \to E$.*

1. *There is a class $U \in H^k(E, E_0; \mathbb{Z}_2)$, so that for all $b \in B$, $\mathrm{inc}^*\mathrm{U} \in \mathrm{H}^k(\mathrm{p}^{-1}\mathrm{b}, \mathrm{p}_0^{-1}\mathrm{b}; \mathbb{Z}_2) \cong \mathbb{Z}_2$ is non-zero. Furthermore*

$$\begin{aligned} \phi : H^i(B; \mathbb{Z}_2) &\to H^{i+k}(E, E_0; \mathbb{Z}_2) \\ \alpha &\mapsto p^*\alpha \cup U \end{aligned}$$

   is an isomorphism for all i.

2. *If there is a class $U \in H^k(E, E_0)$, so that for all $b \in B$, $\mathrm{inc}^*\mathrm{U} \in \mathrm{H}^k(\mathrm{p}^{-1}\mathrm{b}, \mathrm{p}_0^{-1}\mathrm{b}) \cong \mathbb{Z}$ is a generator, then*

$$\begin{aligned} \phi : H^i(B) &\to H^{i+k}(E, E_0) \\ \alpha &\mapsto p^*\alpha \cup U \end{aligned}$$

   is an isomorphism for all i.

Proof. For any quasifibration $f : X \to Y$ and point $y \in Y$, there is a Serre spectral sequence

$$E_2^{i,j} = H^i(Y; H^j(f^{-1}y)) \Longrightarrow H^{i+j}X$$

given by the Serre spectral sequence of the associated fibration π_f. If f were a fibration to begin with, there are, a priori, two different Serre spectral sequences, since f can be considered as a quasifibration or as a fibration. They coincide, since if f is a fibration then f and π_f have the same fiber homotopy type.

The collapsing of the relative Serre spectral sequence

$$E_2^{i,j} = H^i(B; H^j(p^{-1}b, p_0^{-1}b; \mathbb{Z}_2) \Longrightarrow H^{i+j}(E, E_0; \mathbb{Z}_2)$$

gives the Thom isomorphism, and the Thom class U is the image of 1 under the Thom isomorphism.

With integer coefficients, the same argument applies except that the E^2-term might have twisted coefficients. However the existence of an integral Thom class in (2) guarantees that the coefficients are untwisted (look at $E_2^{0,k}$). □

Definition 4.1. U is called the **Thom class** and ϕ is called the **Thom isomorphism**. In case 2 above, the spherical (quasi)-fibration is called **orientable** and a choice of Thom class $U \in H^k(E, E_0)$ is called an **orientation**.

Definition 4.2. Suppose $\xi = (p_0 : E_0 \to B)$ is either a vector bundle with the 0-section deleted, a combinatorial sphere bundle, or a spherical (quasi)-fibration. In the three cases respectively, let $p : E \to B$ be the vector bundle, the combinatorial disk bundle, or the obvious map $p : E \to B$ from the mapping cylinder E of p_0. Then the **i-th Stiefel-Whitney class of** ξ is

$$w_i(\xi) = \phi^{-1} Sq^i \phi(1) \in H^i(B; \mathbb{Z}_2).$$

If p_0 is oriented with Thom class $U \in H^k(E, E_0)$, the **Euler class**

$$e(\xi) \in H^k(B)$$

is the image of the Thom class under

$$H^k(E, E_0) \to H^k E \cong H^k B.$$

We next wish to show that Stiefel-Whitney classes and Euler classes satisfy the axioms and the usual properties, but first we had better make clear what is meant by Whitney sum.

Definition 4.3. If $\xi_1 = (B, \mathcal{M}_1 : B \to \mathrm{MacP}(i, E_1))$ and $\xi_2 = (B, \mathcal{M}_2 : B \to \mathrm{MacP}(j, E_2))$ are two matroid bundles, then the **Whitney sum** $\xi_1 \oplus \xi_2$ is the matroid bundle $(B, \mathcal{M}_1 \oplus \mathcal{M}_2 : B \to \mathrm{MacP}(i + j, E_1 \amalg E_2))$ sending each b to the direct sum $\mathcal{M}_1(b) \oplus \mathcal{M}_2(b)$. If $\xi_1 = (p_1 : E_1 \to B)$ and $\xi_2 = (p_2 : E_2 \to B)$ are two spherical (quasi)-fibrations then the **Whitney sum** is

$$\xi_1 \oplus \xi_2 = (p_1 *_B p_2 : E_1 *_B E_2 \to B),$$

where

$$E_1 *_B E_2 = \{[e_0, e_1, t] \in E_1 * E_2 : p_0(e_0) = p_1(e_1) \text{ whenever } t \neq 0, 1\}$$

is the fiberwise join.

It is not difficult to show that the geometric realization of the combinatorial sphere bundle of a Whitney sum of matroid bundles is the Whitney sum of the resulting spherical quasifibrations.

Proposition 4.4. *The four axioms for Stiefel-Whitney classes are satisfied for vector bundles, matroid bundles, and for spherical (quasi)-fibrations.*

Proof. It suffices to prove the axioms hold for spherical fibrations. Axiom 1 holds since $Sq^0 = \mathrm{Id}$ and Sq^i is zero on H^k for $i > k$. Axiom 2 is clear by construction.

The Whitney sum formula (Axiom 3) holds since the Thom class $\phi(1)$ for the fiberwise join is the external product of the Thom classes of the two summands and there is a sum formula for the Steenrod squares.

As for Axiom 4, one may compute w_1 by restricting the canonical line bundle to the circle. Here the bundle is the Möbius strip, which has non-trivial w_1 by direct computation. □

Remark 4.5. While the axioms characterize the Stiefel-Whitney classes of vector bundles (due to the splitting principle), there is no assertion that the axioms give a characterization for the other categories of bundles.

We next show that $w_1(\xi) = 0$ if and only if ξ is orientable.

The oriented MacPhersonian $\mathcal{O}\,\mathrm{MacP}(k,n)$ is defined in [And98]. The elements of the poset $\mathcal{O}\,\mathrm{MacP}(k,n)$ are all chirotopes of elements of $\mathrm{MacP}(k,n)$. In [And98] it is shown that $\|\mathcal{O}\,\mathrm{MacP}(k,n)\|$ is the universal double cover of $\|\mathrm{MacP}(k,n)\|$. One can also define $\mathcal{O}\,\mathrm{MacP}(k,\infty)$ and show that its geometric realization is the double cover of $\|\mathrm{MacP}(k,\infty)\|$.

Definition 4.6. An **orientation** of a matroid bundle $\xi = (B, \mathcal{M})$ is a poset lifting

$$
\begin{array}{ccc}
 & & \mathcal{O}\,\mathrm{MacP}(k,\infty) \\
 & \nearrow & \downarrow \\
B & \stackrel{\mathcal{M}}{\rightarrow} & \mathrm{MacP}(k,\infty).
\end{array}
$$

Proposition 4.7. *Any topological lifting of* $\|\mathcal{M}\| : \|B\| \rightarrow \|\mathrm{MacP}(k,\infty)\|$ *to* $\|\mathcal{O}\,\mathrm{MacP}(k,\infty)\|$ *is the geometric realization of an orientation of* $\mathcal{M} : B \rightarrow \mathrm{MacP}(k,\infty)$.

Proof. Any topological lifting is simplicial, and any simplicial lifting to a poset covering space is the realization of a poset lifting. (This is clear from looking at the lifting on individual simplices.) □

Thus an orientation of a matroid bundle is equivalent to an orientation of the geometric realization of the associated combinatorial sphere bundle.

Theorem 4.8. *Let* ξ *be a vector bundle, a matroid bundle, or a spherical (quasi)-fibration. Then* ξ *is orientable if and only if* $w_1(\xi) = 0$.

Proof. Suppose first that $\xi = (B, \mathcal{M})$ is a matroid bundle. From the double cover result, $H^1(\|\mathrm{MacP}(k,\infty)\|) \cong \mathbb{Z}_2$, and is generated by $w_1(\gamma_k)$ (where γ_k is the universal bundle) since the first Stiefel-Whitney class is a non-trivial characteristic class.

Note that the map $\|\mathcal{O}\,\mathrm{MacP}(k,\infty)\| \rightarrow \|\mathrm{MacP}(k,\infty)\|$ is an S^0 bundle, and hence has a classifying map into $\mathbb{R}P^\infty$. Thus we have maps

$$
\begin{array}{ccccc}
 & & \|\mathcal{O}\,\mathrm{MacP}(k,\infty)\| & \rightarrow & S^\infty \\
 & & \downarrow & & \downarrow \\
\|B\| & \stackrel{\|\mathcal{M}\|}{\rightarrow} & \|\mathrm{MacP}(k,\infty)\| & \stackrel{c}{\rightarrow} & \mathbb{R}P^\infty
\end{array}
$$

Let $\beta : \|B\| \to \mathbb{R}P^\infty$ be the composition of the lower two maps, ω be the generator of $H^1(\mathbb{R}P^\infty;\mathbb{Z}_2)$. Then by covering space theory β has a lifting if and only if $\beta^*\omega = 0$. One can see directly that the combinatorial vector bundle corresponding to the Möbius strip mapping to the circle is non-orientable, and thus $c^*\omega \neq 0$. Hence $c^*\omega = w_1(\gamma_k)$, and the result follows.

In the other cases, it suffices to consider a spherical fibration. One could either use the classifying spaces $BSG(k)$ and $BG(k)$ for (oriented) spherical fibrations and proceed as above, or use the fact the Sq^1 is the mod 2 Bockstein to show that $w_1(\xi) = 0$ if and only if the coefficients in the spectral sequence used in the Thom isomorphism theorem are untwisted.

□

Finally, we note that proof of the Whitney sum formula for matroid bundles also shows:

Proposition 4.9. *Let ξ_1 and ξ_2 be matroid bundles with orientations. Then*

$$e(\xi_1 \oplus \xi_2) = e(\xi_1) \cup e(\xi_2).$$

In particular, the Euler class is an unstable characteristic class. Indeed, if ϵ is a trivial ($\mathcal{M}$ is constant) bundle of rank greater than zero, then $e(\xi \oplus \epsilon) = e(\xi) \cup 0 = 0$.

5 Vector bundles vs. matroid bundles: the Comparison Theorem

Comparison Theorem. *Let B be a regular cell complex. The composite of the natural transformations*

$$V_k(B) \xrightarrow{C} M_k(B) \xrightarrow{\|E_0\|} Q_k(B)$$

coincides with the forgetful map given by deleting the zero section of a vector bundle.

Thus the Stiefel-Whitney classes of the combinatorialization of a real vector bundle coincide with those of the original bundle. In particular, as a corollary we have Theorem A. In addition, since for every realized rank n oriented matroid M the map $G(k,\mathbb{R}^n) \to \|\operatorname{MacP}(k,\infty)\|$ factors as

$$G(k,\mathbb{R}^n) \to \|\Gamma(k,M)\| \to \|\operatorname{MacP}(k,\infty)\|,$$

and since $G(k,\mathbb{R}^n) \to G(k,\mathbb{R}^\infty)$ gives a split surjection on mod 2 cohomology, we have Theorem B.

Remark 5.1. The Comparison Theorem could also be stated universally by saying that there are maps

$$BO(k) \to \|\operatorname{MacP}(k,\infty)\| \to BG(k)$$

covered by maps of spherical quasifibrations on the universal sphere bundles.

Let M be a rank n oriented matroid realized by a collection $\{\phi_1, \ldots, \phi_m\}$ of linear forms on $\mathbb{R}^n$. Let $S(k, \mathbb{R}^n)$ be the sphere bundle of the canonical bundle over $G(k, \mathbb{R}^n)$. An element of $S(k, \mathbb{R}^n)$ is a pair (V, p) where V is a k-plane in $\mathbb{R}^n$ and $p \in V$ has unit length. Let $E_0(k, M)$ be the combinatorial sphere bundle of the canonical bundle over $\Gamma(k, M)$. Define $\nu(V, p) = (\mu(V), X(p)) \in E_0(k, M)$, where $\mu : G(k, \mathbb{R}^n) \to \Gamma(k, M)$ is the function defined in Section 1.3, $X(p) = (\text{sign } \phi_1(p), \ldots, \text{sign } \phi_m(p))$ is a sign vector. Thus we have a commutative diagram

$$\begin{array}{ccc} S(k, \mathbb{R}^n) & \xrightarrow{\nu} & E_0(k, M) \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle \pi} \\ G(k, \mathbb{R}^n) & \xrightarrow{\mu} & \Gamma(k, M) \end{array}$$

with μ and ν upper semi-continuous.

Lemma 5.2. *Let M be a realized rank n oriented matroid. Then there is a homotopy commutative diagram of continuous maps*

$$\begin{array}{ccc} S(k, \mathbb{R}^n) & \xrightarrow{\tilde{\nu}} & \|E_0(k, M)\| \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle \|\pi\|} \\ G(k, \mathbb{R}^n) & \xrightarrow{\tilde{\mu}} & \|\Gamma(k, M)\| \end{array}$$

so that there is a $V \in G(k, \mathbb{R}^n)$ so that $\tilde{\nu}$ gives a homotopy equivalence from the fiber above V to the fiber above $\tilde{\mu}(V)$. Furthermore $\tilde{\mu}$ is the map specified by Lemma 2.10.

Proof. We again use Appendix A and the semi-algebraic triangulation theorem of [Hir75]. As in the proof of Lemma 2.10, there is a semi-algebraic triangulation $T_G : \|L\| \to G(k, \mathbb{R}^n)$ refining the stratification of $G(k, \mathbb{R}^n)$ and a map $\tilde{\mu} : G(k, \mathbb{R}^n) \to \|\Gamma(k, M)\|$ so that $\tilde{\mu} \circ T_G$ is simplicial. Now consider the semi-algebraic stratification of $S(k, \mathbb{R}^n)$ given by the intersections of the preimages of elements under ν and the preimages of simplices under $\Delta T_G^{-1} \circ p$. By [Hir75], there is a triangulation $T_S : \|K\| \to S(k, \mathbb{R}^n)$ refining this stratification, and so by Corollary A.4 there is a map $\tilde{\nu} : S(k, \mathbb{R}^n) \to \|E_0(k, M)\|$ so that $\tilde{\nu} \circ \Delta T_S$ is simplicial.

To see that these maps make the above diagram commute up to homotopy, let $s \in S(k, \mathbb{R}^n)$. Then s lies in a simplex $\kappa \subset S(k, \mathbb{R}^n)$ of ΔT_S and there is a simplex λ of ΔT_G so that $p(\kappa) \subseteq \lambda \subset G(k, \mathbb{R}^n)$. By construction $\tilde{\nu}(\kappa)$ is contained in the simplex spanned by the totally ordered set $\nu(\kappa)$ and $\tilde{\mu}(\lambda)$ is contained in the simplex spanned by the totally ordered set $\mu(\lambda)$. Then $\tilde{\mu}(p(s))$ and $\|\pi\|(\tilde{\nu}(s))$ both lie in the closed simplex spanned by $\mu(\lambda)$, so there is a straight-line homotopy from $\|\pi\| \circ \tilde{\nu}$ to $\tilde{\mu} \circ p$.

Finally, note that for every vertex $V \in G(k, \mathbb{R}^n)$ of T_G, the topological realization theorem gives a homotopy equivalence from the fiber over V to the fiber over $\tilde{\mu}(V) = \mu(V)$. $\square$

Lemma 5.3. *There is a homotopy commutative diagram*

$$\begin{array}{ccc} S(k,\mathbb{R}^\infty) & \xrightarrow{\tilde{\nu}} & \|E_0(k,\infty)\| \\ \Big\downarrow p & & \Big\downarrow \|\pi\| \\ G(k,\mathbb{R}^\infty) & \xrightarrow{\tilde{\mu}} & \|\operatorname{MacP}(k,\infty)\| \end{array}$$

so that there is a $V \in G(k,\mathbb{R}^\infty)$ so that $\tilde{\nu}$ gives a homotopy equivalence from the fiber above V to the fiber above $\tilde{\mu}(V)$. Furthermore $\tilde{\mu}$ is in the homotopy class of maps specified by Theorem 2.12.

Proof. The proof uses Theorem A.5 and the techniques of the proof of the previous lemma. ☐

Proof of the Comparison Theorem. Let $\xi = (p : E \to B)$ be a vector bundle. Convert the map $\|\pi\|$ to a fibration $\pi_{\|\pi\|}$ and consider the diagram

$$\begin{array}{ccccccc} E_0(\xi) & \longrightarrow & S(k,\mathbb{R}^\infty) & \xrightarrow{\tilde{\nu}} & \|E_0(k,\infty)\| & \xrightarrow{h} & P_{\|\pi\|} \\ \Big\downarrow & & \Big\downarrow p & & \Big\downarrow \|\pi\| & & \Big\downarrow \pi_{\|\pi\|} \\ B & \xrightarrow{c} & G(k,\mathbb{R}^\infty) & \xrightarrow{\tilde{\mu}} & \|\operatorname{MacP}(k,\infty)\| & \xrightarrow{\mathrm{Id}} & \|\operatorname{MacP}(k,\infty)\| \end{array}$$

By the homotopy lifting property, there is a map $\tilde{\nu}' \simeq h\circ\tilde{\nu}$ so that $\pi_{\|\pi\|}\circ\tilde{\nu}' = \tilde{\mu}\circ p$. Furthermore, h gives a homotopy equivalence on fibers (since $\|\pi\|$ is a quasifibration) and $\tilde{\nu}$ gives a homotopy equivalence on a fiber, so $\tilde{\nu}'$ gives a homotopy equivalence on fibers.

Recall $C[\xi]$ is defined by applying the Simplicial Approximation Theorem to find a subdivision B' of B and a map $c' : \mathcal{F}(B') \to \operatorname{MacP}(k,\mathbb{R}^\infty)$, where $\|c'\|$ is homotopic to $\tilde{\mu}\circ c$. Then $C[\xi] = [c']$.

We then have the following equations in $Q_k(B)$:

$$\begin{aligned} \|E_0\|\circ C[\xi] &= \|E_0\|[c'] \\ &= [\|E_0\|{c'}^*(\gamma^k)] \\ &= [\|{c'}^*E_0(\gamma^k)\|] \\ &= [\|c'\|^*\|E_0(\gamma^k)\|] \\ &= [c^*\tilde{\mu}^*\|E_0(\gamma^k)\|] \\ &= [c^*\tilde{\mu}^*P_{\|\pi\|}] \\ &= [c^*S(k,\mathbb{R}^\infty)] \\ &= [E_0(\xi)] \end{aligned}$$

☐

6 Homotopy groups of the combinatorial Grassmannian

One can use the classical J-homomorphism to obtain limited information about $\pi_i\|\operatorname{MacP}(k,n)\|$, or more generally about the homotopy groups of $\|\Gamma(k,M^n)\|$ for realizable M^n. The idea is use vector bundles over spheres to construct elements and use homotopy groups of spheres and characteristic classes to detect them.

A few remarks will give a context for these results. The duality theorem for oriented matroids gives $\|\operatorname{MacP}(k,n)\| \cong \|\operatorname{MacP}(n-k,n)\|$. It is easy to show $\|\operatorname{MacP}(k,n)\|$ is connected, but there exist examples of M^n such that $\|\Gamma(n-1,M^n)\|$ is disconnected ([MRG93]). In [And98], it was shown that $\pi_1\|\operatorname{MacP}(k,n)\| \cong \pi_1 G(k,\mathbb{R}^n)$, and stability results for large n were established. There are also some results on homotopy type of combinatorial Grassmannians for small values of k or n and for oriented matroids with few elements (cf. [MZ93], [Bab93], [SZ93]).

See [Whi78] for the definition of the J-homomorphism $J_{i,k} : \pi_i O(k) \to \pi_{i+k}S^k$. The limit as $k \to \infty$ is denoted

$$J_i : \pi_i O \to \pi_i^S.$$

Stability results for the domain and range of J show

$$\begin{array}{ll} \operatorname{Im} J_{i,k} \to \operatorname{Im} J_{i,k+1} & \text{is an epimorphism if } k \geq i+1 \text{ and} \\ \operatorname{Im} J_{i,k} \to \operatorname{Im} J_i & \text{is an isomorphism if } k > i+1. \end{array}$$

A group H is a **subquotient** of a group G if H is isomorphic to a subgroup of a quotient group of G.

Theorem 6.1. *Let M^n be a realized rank n oriented matroid. Let p be a point in the image of $\tilde{\mu} : G(k,\mathbb{R}^n) \to \|\Gamma(k,M^n)\|$.*

1. $\operatorname{Im} J_{i-1,k}$ *is a subquotient of* $\pi_i(\|\Gamma(k,M^n)\|,p)$ *when* $n-k \geq i$.

2. $\operatorname{Im} J_{i-1}$ *is a subquotient of* $\pi_i(\|\Gamma(k,M^n)\|,p)$ *when* $n-k \geq i$ *and* $k > i$.

Proof. Let $G(k)$ denote the monoid of self-homotopy equivalences of S^{k-1}, given the compact-open topology. Its classifying space $BG(k)$ classifies rank k spherical (quasi)-fibrations [Sta63]. $G_0(k+1)$ denotes the monoid of self-homotopy equivalences of S^k which fix a base point $*$.

The result follows from commutativity up to homotopy of the diagram

$$\begin{array}{ccccc} G(k,\mathbb{R}^n) & \xrightarrow{\tilde{\mu}} & \|\Gamma(k,M^n)\| & & \\ \downarrow \beta & & \downarrow \delta & & \\ G(k,\mathbb{R}^\infty) & \xrightarrow{\gamma} & BG(k) & \xrightarrow{\epsilon} & BG_0(k+1), \end{array}$$

the surjectivity of $\pi_i(\beta)$ when $n-k \geq i$, and the identification of $\pi_i(\epsilon \circ \gamma)$ with $J_{i-1,k}$. The map β is given by inclusion, the map γ by classifying the

canonical bundle minus its zero section, and the map ϵ is B applied to the injection of monoids $G(k) \to G_0(k)$ given by suspension. The map $\tilde{\mu}$, the map δ, and the homotopy $\delta \circ \tilde{\mu} \simeq \gamma \circ \beta$, are given by our three main theorems: the Combinatorialization Theorem, Spherical Quasifibration Theorem, and the Comparison Theorem (see also Lemma 5.2).

The surjectivity of $\pi_i(\beta)$ when $n-k \geq i$ follows either from the Cellular Approximation Theorem applied to the Schubert cell decomposition, or by viewing $G(k,n)$ as $O(n)/(O(k) \times O(n-k))$ and using the homotopy long exact sequence of a fibration.

Now $\pi_i G(k, \mathbb{R}^\infty) \cong \pi_{i-1} O(k)$ since there is a fibration $O(k) \to V(k, \mathbb{R}^\infty) \to G(k, \mathbb{R}^\infty)$ and the Stiefel manifold is contractible. Also $\pi_i BG_0(k+1) \cong \pi_i G_0(k+1)$, which is in turn $\pi_{i+k} S^k$ by the adjoint property of smash and mapping spaces in the category of based CW complexes. Thus we have identified the domain and range of $\pi_i(\epsilon \circ \gamma)$ with that of $J_{i-1,k}$, and the identification of the two maps consists of tracing through these identifications. □

Corollary 6.2. *Let M^n be a realized rank n oriented matroid. Let p be a point in the image of $\tilde{\mu} : G(k, \mathbb{R}^n) \to \|\Gamma(k, M^n)\|$.*

1.

$$\pi_2(\| \operatorname{MacP}(k,n)\|) \cong \pi_2(G(k,\mathbb{R}^n)) \cong \begin{cases} 0 & \text{if } k=1 \\ \mathbb{Z} & \text{if } k=2 \text{ and } n-k \geq 2 \\ \mathbb{Z}_2 & \text{if } k \geq 3 \text{ and } n-k \geq 3. \end{cases}$$

2. *$\pi_k(\|\Gamma(k, M^n)\|, p)$ has $\mathbb{Z}$ as a subgroup when k is even and $n \geq 2k$.*

3. *$\pi_i(\|\Gamma(k, M^n)\|, p)$ has $\mathbb{Z}_2$ as a subquotient when $i \equiv 1, 2$ (mod 8), $n-k \geq i$, and $k \geq i$.*

4. *$\pi_{4m}(\|\Gamma(k, M^n)\|, p)$ has $\mathbb{Z}_{a_m}$ as a subquotient when $m > 0$, $n-k \geq 4m$, and $k \geq 4m$. Here a_m is the denominator of $B_m/4m$ expressed as a fraction in lowest terms, and B_m is the m-th Bernoulli number.*

Proof. 1. By [And98], the combinatorialization map $\pi_2(G(k, \mathbb{R}^n)) \to \pi_2(\| \operatorname{MacP}(k,n)\|)$ is surjective, so (1) follows from (2) and (3). An alternate route to (1) is to use characteristic classes to detect elements of $\pi_2 \| \operatorname{MacP}(k,n)\|$, by applying the Euler class and second Stiefel-Whitney class of the combinatorialization of the complex Hopf bundle over the 2-sphere.

2. There is a characteristic class version and a homotopy theoretic version of the proof. The characteristic class proof is as follows. Consider the oriented combinatorial Grassmannian $\tilde{\Gamma}(k, M^n)$, defined analogously to $\mathcal{O} \operatorname{MacP}(k,n)$. The forgetful map $\|\tilde{\Gamma}(k, M^n)\| \to \|\Gamma(k, M^n)\|$ is a double cover, so $\pi_k(\|\Gamma(k, M^n)\|, p) \cong \pi_k(\|\tilde{\Gamma}(k, M^n)\|, p')$ for each lifting p' of p. The tangent bundle of the k-sphere combinatorializes to a map $S^k \to \tilde{G}(k, \mathbb{R}^n) \xrightarrow{\tilde{\mu}} \|\tilde{\Gamma}(k, M^n)\|$. The evaluation of the Euler class of the tangent bundle of a manifold on the fundamental class of

that manifold is the Euler characteristic of the manifold (cf. 11.12 in [MS74]). In particular, the Euler class of the tangent bundle of an even-dimensional sphere represents twice the generator of $H^k(S^k) \cong \mathbb{Z}$. The Euler class applied to oriented vector bundles over k-spheres can be viewed as a homomorphism from $\pi_k(BSG(k))$ to $H^k(S^k)$, where $BSG(k)$ classifies rank k oriented spherical fibrations. Thus the combinatorialization of this tangent bundle generates an infinite subgroup of $\pi_k(\|\tilde{\Gamma}(k, M^n)\|, p'))$.

The homotopy theoretic version is to consider the Hopf invariant

$$H : \operatorname{Im} J_{k-1,k} \to \mathbb{Z}.$$

The domain of $J_{k-1,k}$ is $\pi_{k-1}(O(k))$, which classifies k-bundles over S^k. Now $H \circ J_{k-1,k}$ is the Euler class, so $2\mathbb{Z} \subseteq \operatorname{Im} H$, using the tangent bundle of the k-sphere again. (For the reader's edification, we note the Hopf invariant is onto if and only if $k = 2, 4$ or 8, as can be seen by Bott periodicity or by Adams's work on Hopf invariant one.)

3.,4. This follows from Theorem 6.1 and the deep homotopy theoretic computation of $\operatorname{Im} J$ due to Adams [Ada65]. The result is that $\operatorname{Im} J_{i-1}$ is $\mathbb{Z}_2$ for $i \equiv 1, 2 \pmod 8$, $\mathbb{Z}_{a_m}$ for $i = 4m$, and is zero otherwise. $\square$

An exposition of Bernoulli numbers and topology is given in [MS74, Appendix B]. The first few values of a_m are :

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
24	240	504	480	264	65520	24	16320

By the stability results of [And98], the results of our corollary also apply to the MacPhersonian $\operatorname{MacP}(k, \infty)$.

7 Vector fields and characteristic classes

The classical motivation for characteristic classes was as obstructions to the existence of linearly independent vector fields of a manifold, or more generally, independent sections of a vector bundle. This section gives a combinatorial analog.

Definition 7.1. The **combinatorial Stiefel manifold** $V_l(k, n)$ is the poset of all $M \in \operatorname{MacP}(k, n + l)$ satisfying both of the conditions below

1. $\operatorname{rank}(M \backslash \{n + 1, \ldots, n + l\}) = k$
2. $\{n + 1, \ldots, n + l\}$ is independent in M.

Note that deleting $\{n + 1, \ldots, n + l\}$ gives a surjective map $V_l(k, n) \to \operatorname{MacP}(k, n)$ if $l \leq k$.

Definition 7.2. If (B, ξ) is a matroid bundle, an **independent set of l vector fields** is a lifting

$$\begin{tikzcd} & V_l(k,n) \arrow[d] \\ B \arrow[ur] \arrow[r, "\xi"] & \operatorname{MacP}(k,n). \end{tikzcd}$$

Lemma 7.3. *If a matroid bundle $\xi = (B, \mathcal{M} : B \to \mathrm{MacP}(k, n)$ admits a set $\nu : B \to V_l(k, n)$ of l independent vector fields, then:*

1. *The map*
$$\begin{array}{rccl} \mathcal{Q}: & B & \to & \mathrm{MacP}(k-l, n) \\ & \sigma & \mapsto & \nu(\sigma)/\{n+1, \ldots, n+l\} \end{array}$$
is a matroid bundle.

2. *If ϵ_l is the trivial rank l bundle over B sending each cell to the rank l oriented matroid with elements $\{n+1, \ldots, n+l\}$, then the matroid bundles ξ and $\mathcal{Q} \oplus \epsilon_l$ are $\|B\|$-isomorphic.*

$\mathcal{Q}$ is called the **quotient bundle** of ν.

Proof. (1) This follows immediately from Lemma 3.7

(2) The proof is by induction on l, and relies on the Order Homotopy Lemma.

If $l = 1$, first note that $\nu(\sigma) \geq \xi(\sigma)$ for all σ, so $\nu \simeq \xi$. (Here $\simeq$ means $\|B\|$-isomorphic. All homotopies occur in $\|\mathrm{MacP}(k, n+l)\|$; there is no need to go to ∞.) Thus, it suffices to show that $\nu(\sigma) \geq \mathcal{Q}(\sigma) \oplus \epsilon_1(\sigma)$ for all σ. Any covector $X \times Y \in \mathcal{V}^*(\mathcal{Q}(\sigma) \oplus \epsilon_1(\sigma))$ is built from covectors $X \in \mathcal{V}^*(\mathcal{Q}(\sigma)) \subset \mathcal{V}^*(\nu(\sigma))$ and $Y : \{n+1\} \to \{+, -, 0\}$. Since $\{n+1\}$ is independent in $\nu(\sigma)$, there is a $\widetilde{Y} \in \mathcal{V}^*(\nu(\sigma))$, so that $\widetilde{Y} \geq Y$. Thus $X \circ \widetilde{Y}$ is a covector of $\nu(\sigma)$ which is greater than or equal to $X \times Y$.

For $l > 1$, let ν' be the set of $l-1$ independent vector fields obtained from ν by deleting $n+l$ from each oriented matroid $\nu(\sigma)$, and let $\mathcal{Q}'$ be the resulting quotient bundle. Note that the vector field ν_l on $\mathcal{Q}'$ given by $n+l$ is non-vanishing and $\mathcal{Q}$ is the quotient bundle of $\mathcal{Q}'$ by ν_l. Hence we have

$$\begin{aligned} \xi &\simeq \mathcal{Q}' \oplus \epsilon_{l-1} && \text{by the induction hypothesis} \\ &\simeq \mathcal{Q} \oplus \epsilon_1 \oplus \epsilon_{l-1} && \text{by the } l=1 \text{ case.} \end{aligned}$$

□

Theorem 7.4. *If a rank k matroid bundle ξ admits an independent set of l vector fields, then $w_{k-l+1}(\xi) = 0$.*

Proof of Theorem 7.4. If ν is a set of l independent vector fields in ξ and $\mathcal{Q}$ is the resulting quotient bundle, then by the above lemma, $\xi \simeq \mathcal{Q} \oplus \epsilon_l$. Thus by the Whitney sum formula, $w(\xi) = w(\mathcal{Q})w(\epsilon_l)$ where $w = 1 + w_1 + w_2 + \cdots$ is the total Stiefel-Whitney class. But $w(\epsilon_l) = 1$ since ϵ_l is trivial, and $w_{k-l+1}(\mathcal{Q}) = 0$ since $\mathcal{Q}$ is a rank $k-l$ bundle. □

Similarly, we have

Theorem 7.5. *If a rank k matroid bundle ξ admits a non-zero cross section (i.e. an independent set of 1 vector field) then the Euler class $e(\xi)$ vanishes.*

8 Some open questions

There are open questions everywhere you spit; we list a few.

1. Is a CD manifold a Poincaré complex? Does a CD manifold satisfy Poincaré duality?

2. Give a definition of isomorphism of CD-manifolds; show that a diffeomorphism class of smooth manifolds determines an isomorphism class of CD-manifolds.

   Discussion: Problems (1) and (2) are manifold theoretic analogues of the bundle theoretic results of this paper. Macpherson [Mac93] defined a CD manifold and showed how a smooth manifold with a smooth triangulation determines a CD manifold. He asked whether a CD-manifold was a topological manifold; this was shown in a special case in [And99b].

3. Are there exotic mod 2 characteristic classes?

4. Are there Pontrjagin classes?

   Discussion: The authors together with Eric Babson have outlined a construction of rational Pontrjagin classes. If these classes were integral cohomology classes, that would imply the existence of exotic CD 7-spheres.

5. Is $\mathrm{MacP}(\infty, \infty)$ an infinite loop space?

6. Compute the homotopy groups of $\mathrm{MacP}(\infty, \infty)$.

   Discussion: Solving questions 5 and 6 would show that combinatorial vector bundles give a generalized cohomology theory and compute the coefficient groups (isomorphism classes of matroid bundles over spheres.)

A Topological maps from combinatorial ones

Section 1.3 described a natural map $\mu : G(k, \mathbb{R}^n) \to \Gamma(k, M)$ for any rank n oriented matroid M with a fixed realization in $\mathbb{R}^n$, given by intersecting the hyperplanes of the realization with $V \in G(k, \mathbb{R}^n)$ and taking the corresponding oriented matroid. This appendix describes how we use this map to make a simplicial map from a triangulation of $G(k, \mathbb{R}^n)$ to $\Delta\Gamma(k, M)$, unique up to homotopy. We also construct a topological map $G(k, \mathbb{R}^\infty) \to \|\mathrm{MacP}(k, \infty)\|$ which is in some sense the limit of the simplicial maps obtained when $\Gamma(k, M) = \mathrm{MacP}(k, n)$. As discussed in [AD], this limit map is not PL.

Actually, we will work more generally, considering maps from spaces to posets satisfying certain properties.

Definition A.1. A **triangulation** of a topological space X is a homeomorphism $T : \|K\| \to X$ where K is a simplicial complex. We will abuse language slightly and refer to the image $T(\|\sigma\|)$ of a simplex $\|\sigma\|$ under a triangulation T as a **simplex of** T. If X resp. Y are spaces equipped with triangulations S

resp. T then a map $f : X \to Y$ is **simplicial** if $T^{-1} \circ f \circ S$ is. A **subdivision** of a simplicial complex K is a homeomorphism $S : \|K'\| \to \|K\|$ where K' is a simplicial complex and for every simplex $\sigma' \in K'$, there is a simplex $\sigma \in K$ so that $S(\|\sigma'\|) \subseteq \|\sigma\|$ and S is linear on $\|\sigma'\|$. The triangulation $T' = T \circ S$ is a **subdivision of the triangulation** T. An example of such is the barycentric subdivision $\Delta T : \|\Delta K\| \to X$. Two triangulations of a space X are **equivalent** if they have a common subdivision. A **PL space** is a space equipped with a fixed equivalence class of triangulations, called the **PL triangulations**. A map $f : X \to Y$ between PL spaces is a **PL map** if there are PL triangulations so that f is simplicial.

Definition A.2. Let $\mu : G \to M$ be a function from a space to a poset. The partition $\{\mu^{-1}(m) : m \in M\}$ of G is the **stratification of** G **induced by** μ. The map μ is **upper semi-continuous** if every $g \in G$ has a neighborhood U so that $\mu(U) \subseteq M_{\geq \mu(g)}$. Thus the closure of a stratum $\mu^{-1}(m)$ maps to the lower order ideal $M_{\leq m}$ of M. A triangulation $T : \|K\| \to G$ **refines the stratification** if the interior of every simplex maps under to a single element of the poset M.

Lemma A.3. *Let $\mu : G \to M$ be upper semi-continuous and let $T : \|K\| \to G$ be a triangulation refining the stratification. Then for any simplex α of the barycentric subdivision, $\mu(\Delta T\|\alpha\|)$ is totally ordered.*

Proof. For a simplex σ of K, let $\langle\sigma\rangle$ denote both the barycenter of the simplex $\|\sigma\| \subseteq \|K\|$ and the corresponding vertex of $\|\Delta K\|$. Let $\alpha = \{\sigma_n > \cdots > \sigma_0\} \in \Delta K$ be a chain of simplices of K. Note that

$$\|\alpha\| \subseteq (\text{int } \|\sigma_n\|) \cup \|\{\sigma_{n-1} > \cdots > \sigma_0\}\|$$

hence inductively,

$$\mu(\Delta T\|\alpha\|) = \{\mu(T\langle\sigma_n\rangle), \ldots, \mu(T\langle\sigma_0\rangle)\}.$$

Since the triangulation refines the stratification and μ is upper semi-continuous, $\mu(T\langle\sigma_i\rangle) \geq \mu(T\langle\sigma_{i-1}\rangle)$ for all i. □

Corollary A.4. *Let $\mu : G \to M$ be upper semi-continuous and let $T : \|K\| \to G$ be a triangulation refining the stratification.*

1. *There is a map $\tilde{\mu}_T : G \to \|M\|$, simplicial with respect to the barycentric subdivision of T, which agrees with μ on the vertices of ΔT.*
2. *If T' is a subdivision of T, then $\tilde{\mu}_T \simeq \tilde{\mu}_{T'}$.*

Proof. Part (1) follows from the last lemma by defining $\tilde{\mu}_T$ on vertices and extending by linearity with respect to $\|K\|$. Part (2) follows from a straight-line homotopy $t\tilde{\mu}_T(a) + (1-t)\tilde{\mu}_{T'}(a)$ across the simplices of $\|M\|$. □

We need an infinite version of the corollary.

Theorem A.5. *Let $G_1 \subseteq G_2 \subseteq \cdots$ be a sequence of PL spaces, $M_1 \subseteq M_2 \subseteq \cdots$ a sequence of posets, and $\mu_1 : G_1 \to M_1, \mu_2 : G_2 \to M_2, \ldots$ a sequence of upper semi-continuous maps so that $\mu_i|_{G_{i-1}} = \mu_{i-1}$. Let $T_1, T_2, \ldots$ be a sequence of PL triangulations of $G_1, G_2, \ldots$ refining the stratifications given by $\mu_1, \mu_2, \ldots$, so that the restriction of the i-th triangulation is a subdivision of the $(i-1)$-st triangulation. Let M be the union of the M_i's and G be the direct limit of the G_i's.*

1. *There is a continuous map $\tilde{\mu} : G \to \|M\|$ which, for every i, restricts to a map $\tilde{\mu}_i : G_i \to \|M_i\|$ which is simplicial with respect to the barycentric subdivision of T_i, and so that $\tilde{\mu}_i$ agrees with μ_i on the vertices of ΔT_i in $G_i \backslash G_{i-1}$.*
2. *Let $T_1', T_2', \ldots$ be another sequence of PL triangulations satisfying the same hypotheses as $T_1, T_2, \ldots$. Let $\tilde{\mu}' : G \to \|M\|$ be the map given by part (1) using the T_i''s. Then $\tilde{\mu} \simeq \tilde{\mu}'$ and the homotopy restricts to a homotopy $\tilde{\mu}_i \simeq \tilde{\mu'}_i$ for all i.*

Proof. We will inductively define maps $S_i : G_i \to \Delta M_i$ so that

1. $S_i|_{G_{i-1}} = S_{i-1}$,
2. For a vertex $v \in G_i \backslash G_{i-1}$ of ΔT_i,
$$S_i(v) = \{\mu(v)\},$$
3. For a point $p \in G_i$ in the interior of a simplex of ΔT_i which is spanned by vertices $\{v_0, \ldots, v_n\} \subset G_i$,
$$S_i(p) = \cup_j S_i(v_j).$$

Assume inductively that S_{i-1} has been defined satisfying (1), (2), and (3), and also assume inductively that S_{i-1} satisfies property

4. $\max S_{i-1}(p) = \mu(p) \qquad$ for $p \in G_{i-1}$.

We then use property (1) to define S_i on G_{i-1} and properties (2) and (3) to define S_i on $G_i \backslash G_{i-1}$. We need to verify three things: first that property (3) continues to hold for S_i and $p \in G_{i-1}$, second that $S_i(p)$ is totally ordered for $p \in G_i \backslash G_{i-1}$, and third that property (4) holds. We leave the verification of the first part to the reader.

We now show that $S_i(p)$ is totally ordered for $p \in G_i \backslash G_{i-1}$. Suppose p is in the interior of a simplex of ΔT_i with vertices

$$\{v_0, \ldots, v_n\} \cup \{w_0, \ldots, w_m\}$$

with the v's in G_{i-1} and the w's in $G_i \backslash G_{i-1}$. Then by choosing a point q in the interior of the simplex spanned by the v's and by the (omitted) proof that

property (3) holds for S_i and q, we see $\cup_j S_i(v_j)$ is totally ordered. The proof of Lemma A.3 shows that $\cup_j S_i(w_j) = \{\mu(w_0), \dots, \mu(w_m)\}$ is totally ordered. Because G_{i-1} is a subcomplex of the triangulation T_i, because we have taken a barycentric subdivision, because μ is upper semi-continuous, and because T_i refines the stratification, $\mu(w_j) \geq \mu(v_k)$ for all j and k. Hence $S_i(p)$ is totally ordered. Finally property (4) holds since we have taken a barycentric subdivision.

Next we use S_i to define $\tilde{\mu}_i$ inductively. For $p \in G_{i-1}$, let $\tilde{\mu}_i(p) = \tilde{\mu}_{i-1}(p)$. For a vertex $p \in G_i \backslash G_{i-1}$ of ΔT_i, define $\tilde{\mu}_i(p) = \mu(p)$. Then for p in the interior of a simplex spanned by $\{v_0, \dots, v_k\}$, define $\tilde{\mu}_i(p)$ by linearity, noting inductively that $\mu_i(p)$ is in the closed simplex spanned by $S_i(p)$. This completes the proof of part (1) of the Theorem.

For part (2), it suffices to consider the case when each T_i' subdivides T_i. Then note that $S_i'(p) \subseteq S_i(p)$, so we can use the straight-line homotopy across the simplex spanned by $S_i(p)$.

□

B Babson's criterion

This appendix gives the criterion of Babson for the realization of a poset map to be a quasifibration. We first state two results of Quillen [Qui73, page 98], specialized from general categories to posets.

Quillen's Theorem A. *Let $f : P \to Q$ be a poset map. If for all $q \in Q$, $\|f^{-1}(Q_{\geq q})\|$ is contractible, then $\|f\| : \|P\| \to \|Q\|$ is a homotopy equivalence.*

A commutative square of spaces

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \beta\downarrow & & \downarrow\gamma \\ C & \xrightarrow{\delta} & D \end{array}$$

is **homotopy cartesian** if for all $c \in C$, the induced map on homotopy fibers

$$\pi_\beta^{-1}(c) \to \pi_\gamma^{-1}(\delta c)$$

is a homotopy equivalence.

Quillen's Theorem B. *Let $f : P \to Q$ be a poset map such that for every inequality $q \geq q'$ in Q, the inclusion map $\|f^{-1}(Q_{\geq q})\| \to \|f^{-1}(Q_{\geq q'})\|$ is a homotopy equivalence. Then for any $q \in Q$,*

$$\begin{array}{ccc} \|f^{-1}(Q_{\geq q})\| & \longrightarrow & \|P\| \\ \downarrow & & \downarrow \\ \|Q_{\geq q}\| & \longrightarrow & \|Q\| \end{array}$$

is homotopy cartesian.

Since the geometric realization of a poset can be identified with the realization of the poset obtained by reversing the inequalities, one may reverse the inequalities in Quillen's Theorem A and B. Similar remarks apply to the various lemmas below.

In Babson's thesis, both of the previous two results of Quillen were recast.

Lemma B.1. *If $f : P \to Q$ is a poset map satisfying both of the conditions below, then $\|f\|$ is a homotopy equivalence.*

1. *$\|f^{-1}q\|$ is contractible for all $q \in Q$.*
2. *$\|f^{-1}q \cap P_{\leq p}\|$ is contractible whenever $p \in P$, $q \in Q$, and $q \leq f(p)$.*

Proof. By Theorem A applied to f and condition 1, we only need show that the realization of the inclusion $i : f^{-1}q \to f^{-1}(Q_{\geq q})$ is a homotopy equivalence for all $q \in Q$. But this follows from condition 2 and by applying Theorem A to i, noting that

$$i^{-1}(f^{-1}(Q_{\geq q})_{\leq p}) = f^{-1}q \cap P_{\leq p}$$

□

Recall that a chain in a poset P is a non-empty, totally ordered, finite subset of P, and ΔP denotes the poset of chains in P, where the partial order is given by inclusion. Note $\|\Delta P\|$ is the barycentric subdivision of $\|P\|$. If $f : P \to Q$ is a poset map, then there is a poset map $\Delta f : \Delta P \to \Delta Q$ sending a chain c to the chain $f(d)$. Note that for a chain d of Q, the symbols $f^{-1}d$ and $(\Delta f)^{-1}d$ have different meanings.

Lemma B.2. *If $f : P \to Q$ is a poset map satisfying the conditions below, then $\|f\|$ is a quasifibration.*

1. *$\|f^{-1}q \cap P_{\leq p}\|$ is contractible whenever $p \in P$, $q \in Q$, and $q \leq f(p)$.*
2. *$\|f^{-1}q \cap P_{\geq p}\|$ is contractible whenever $p \in P$, $q \in Q$, and $q \geq f(p)$.*

Proof. We wish to apply Quillen's Theorem B to the induced map of posets $\Delta f : \Delta P \to \Delta Q$. For a chain $d \in \Delta Q$, there is a retraction

$$\begin{aligned} r : (\Delta f)^{-1}(Q_{\geq d}) &\to (\Delta f)^{-1}d \\ c &\mapsto c \cap f^{-1}d. \end{aligned}$$

By the order homotopy lemma, $\|r\|$ is actually a deformation retraction since $r(c) \leq c$ for every $c \in (\Delta f)^{-1}(Q_{\geq d})$.

For an inequality $d \geq d'$ in ΔQ, consider the following commutative diagram:

$$\begin{tikzcd} (\Delta f)^{-1}d \arrow[r] \arrow[d, "\alpha"'] & (\Delta f)^{-1}(Q_{\geq d}) \arrow[d] \\ (\Delta f)^{-1}d' & (\Delta f)^{-1}(Q_{\geq d'}) \arrow[l, "r"'] \end{tikzcd}$$

where $\alpha(c) = c \cap f^{-1}d'$, r is the retraction and the other two arrows are inclusions. To verify the hypothesis of Quillen's Theorem B applied to Δf, it suffices to prove the following lemma.

Lemma B.3. *Let $f : P \to Q$ be a poset map satisfying the conditions of Lemma B.2. Given an inequality $d \geq d'$ in ΔQ, the geometric realization of the map*

$$\begin{aligned} \alpha(d \geq d') : (\Delta f)^{-1} d &\to (\Delta f)^{-1} d' \\ c &\mapsto c \cap f^{-1} d' \end{aligned}$$

is a homotopy equivalence.

Proof. To prove this it suffices to check only those inequalities in ΔQ given by deleting the smallest or largest element of a chain. Let $d = (q_r > \cdots > q_1 > q_0)$ be a chain in ΔQ. Let $d' = (q_r > \cdots > q_2 > q_1)$ and $d'' = (q_{r-1} > \cdots > q_1 > q_0)$. We will use Lemma B.1 to prove that $\|\alpha(d \geq d')\|$ and $\|\alpha(d \geq d'')\|$ are homotopy equivalences. Note $\alpha(d \geq d')^{-1} c'$ is isomorphic to $\Delta(f^{-1} q_0 \cap P_{\leq \min c'})$ by the isomorphism $c \mapsto c \cap f^{-1} q_0$, and the geometric realization of $\Delta(f^{-1} q_0 \cap P_{\leq \min c'})$ is contractible by condition (1).

Now if $c' \subseteq c \cap f^{-1} d'$ where c and c' are chains which map to d and d' respectively,

$$\alpha(d \geq d')^{-1} c' \cap ((\Delta f)^{-1} d)_{\leq c}$$

has a maximum element, namely $(c \cap f^{-1} q_0) \cup c'$, so its geometric realization is contractible. Thus by Lemma B.1, $\|\alpha(d \geq d')\|$ is a homotopy equivalence. The proof that $\|F(d \geq d'')\|$ is a homotopy equivalence is similar and uses property (3) of Lemma B.2. $\square$

Now we return to the proof of Lemma B.2. Note that for $d \in \Delta Q$, the geometric realization of the diagram

$$\begin{array}{ccc} (\Delta f)^{-1} d & \longrightarrow & (\Delta f)^{-1}(Q_{\geq d}) \\ \downarrow & & \downarrow \\ d & \longrightarrow & Q_{\geq d} \end{array}$$

is homotopy cartesian. Thus for all vertices and barycenters of $\|Q\|$ the inclusion of the fiber of $\|f\| : \|P\| \to \|Q\|$ in the homotopy fiber is a homotopy equivalence. Since this is true for the barycenters and since $\|f\|$ is a simplicial map, this is true for all points in the interior of a simplex, so $\|f\|$ is a quasifibration. $\square$

References

[AD] L. Anderson and J. F. Davis. Schubert stratifications and triangulations of the infinite Grassmannian. To appear.

[Ada65] J. F. Adams. On the groups $J(X)$. IV. *Topology*, 5:21–71, 1965.

[And] L. Anderson. Representing weak maps of oriented matroids. To appear in *European Journal of Combinatorics*.

[And98] L. Anderson. Homotopy groups of the combinatorial Grassmannian. *Discrete Comput. Geom.*, 20:549–560, 1998.

[And99a] L. Anderson. Matroid bundles. In *New Perspectives in Algebraic Combinatorics*, MSRI book series. Cambridge University Press, 1999.

[And99b] L. Anderson. Topology of combinatorial differential manifolds. *Topology*, 38(1):197–221, 1999.

[Bab93] E. Babson. *A combinatorial flag space*. PhD thesis, MIT, 1993.

[BLS+93] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G. M. Ziegler. *Oriented matroids*, volume 46 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 1993.

[Dol63] A. Dold. Partitions of unity in the theory of fibrations. *Ann. of Math. (2)*, 78:223–255, 1963.

[DT56] A. Dold and R. Thom. Une généralisation de la notion d'espace fibré. Application aux produits symétriques infinis. *C. R. Acad. Sci. Paris*, 242:1680–1682, 1956.

[FL78] J. Folkman and J. Lawrence. Oriented matroids. *J. Combin. Theory Ser. B*, 25:199–236, 1978.

[GM92] I. M. Gelfand and R. D. MacPherson. A combinatorial formula for the Pontrjagin classes. *Bull. Amer. Math. Soc. (N.S.)*, 26:304–309, 1992.

[Hir75] H. Hironaka. Triangulations of algebraic sets. In *Algebraic geometry (Proc. Sympos. Pure Math., Vol. 29, Humboldt State Univ., Arcata, Calif., 1974)*, pages 165–185. Amer. Math. Soc., 1975.

[Mac93] R. D. MacPherson. Combinatorial differential manifolds. In *Topological methods in modern mathematics (Stony Brook, NY, 1991)*, pages 203–221. Publish or Perish, 1993.

[Mil59] J. Milnor. On spaces having the homotopy type of a CW-complex. *Trans. Amer. Math. Soc.*, 90:272–280, 1959.

[MRG93] N. Mnëv and J. Richter-Gebert. Two constructions of oriented matroids with disconnected extension space. *Disc. and Comp. Geometry*, 10(3):271–286, 1993.

[MS74] J. W. Milnor and J. D. Stasheff. *Characteristic classes*. Number 76 in Annals of Mathematics Studies. Princeton University Press, 1974.

[MZ93] N. Mnëv and G. Ziegler. Combinatorial models for the finite-dimensional Grassmannians. *Discrete Comput. Geom.*, 10(3):241–250, 1993.

[Qui73] D. Quillen. Higher algebraic K-theory, I: Higher K-theories. In *Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972*, number 341 in Lecture Notes in Mathematics, pages 85–147. Springer-Verlag, 1973.

[Sta63] J. Stasheff. A classification theorem for fibre spaces. *Topology*, 2:239–246, 1963.

[SZ93] B. Sturmfels and G. Ziegler. Extension spaces of oriented matroids. *Discrete Comput. Geom.*, 10(1):23–45, 1993.

[Whi78] G. W. Whitehead. *Elements of Homotopy Theory.* Springer-Verlag, 1978.

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